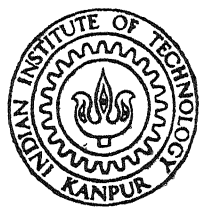
ON OPTIMUM CODES AND THEIR COVERING RADII

by

MADHU SUDAN GARG



DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

November, 1990

ON OPTIMUM CODES AND THEIR COVERING RADII

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

by

MADHU SUDAN GARG

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

November, 1990

CERTIFICATE

It is certified that the work contained in the thesis entitled ''ON OPTIMUM CODES AND THEIR COVERING RADII'' by Mr. Madhu Sudan Garg, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

(M.C. BHANDARI)
Thesis Supervisor
Department of Mathematics
Indian Institute of Technology
Kanpur 208016, U.P.
India

November, 1990

DEDICATED TO MY PARENTS

Shri Prem Chand Garg
Smt. Sarojini Garg

ACKNOWLEDGEMENTS

I am indebted to my thesis supervisor Dr. M.C. Bhandari for having bestowed upon me his valuable guidance and inspiration at all stages of this research. My discussions with him were enlightening.

I have no word to express my gratitude for the affection showered on me by Dr.B.L. Dhooper and Shri Ramashish (social workers).

I am grateful to all my friends who have made this period of study a pleasent and memorable one.

Finally I express my thanks to Mr. Ashok Kumar Bhatia for typing this thesis.

Madhu Sudan Garg

CONTENTS

NOMENCLATURE

q	$\equiv p^m$, where p is a prime and m is a positive integer, p.1.
$GF(q)$	Galois field with q elements, p.1.
$[n,k,d]$	Length, dimension and minimum distance of a linear code over GF(q), p.1.
C	An [n,k,d] code, p.1.
$R(C)$	Covering radius of the code C, p.1.
$\lceil x \rceil$	The smallest integer greater than or equal to x, p.1.
$g_q(k,d)$	$\equiv \sum_{i=0}^{k-1} \lceil d/q^i \rceil$, p.1.
$n_q(k,d)$	The minimal length of a linear code over GF(q) of dimension k and minimum distance d, p.1.
$[n,k]$	Length and dimension of a linear code over GF(q), p.3.
$d_q[n,k]$	The maximal minimum distance of an [n,k] code, p.3.
$k_q[n,d]$	The maximal dimension of a linear code over GF(q) of length n and minimum distance d, p.3.
$GF(q)^n$	The set of all n-tuples over GF(q), p.5.
$wt(x)$	Weight of a vector x,p.5.

$d(x,y)$	Distance between the vectors x and y, p.5.
$d(x,s)$	Distance of a vector x from a set S, p.5.
$H_k(q)$	Hamming code of dimension k over GF(q), p.6.
$S_k(q)$	Simplex code of dimension k over GF(q), p.6.
$G_k(q)$	A generator matrix for $S_k(q)$, p.6.
S_k	Binary Simplex code of dimension k, p.6.
$[A\|B]$	Juxtaposition of matrices A and B, p.7.
$[M \setminus A]$	Matrix obtained by deleting columns of the matrix A from the matrix M, p.7.
$C_{k;u}(q)$	The MacDonald code of dimension k over GF(q), $1 \leq u \leq k-1$, p.7.
$RM(1,k-1)$	First order binary Reed-Muller code of dimension k, p.7.
$C^{\perp}$	Dual of a code C.
A_i	Number of codewords of weight i in a code C, p.8.
B_i	Number of codewords of weight i in $C^{\perp}$, p.8.
$g(k,d)$	$\equiv g_2(k,d)$, p.9.
$n(k,d)$	$\equiv n_2(k,d)$, p.9.
$Res(C,c)$	Residual code of C with respect to the codeword $c \in C$, p.16.
$Res(C,w)$	Residual code of C with respect to the number w, p.16.

$\lfloor x \rfloor$	The largest integer less than or equal to x, p.17.
x^{tr}	Transpose of a vector x, p.23.
$C_{\alpha}^{(i)}$	The set of codewords of a code C, whose ith coordinate is α, p.24.
$H(C) = H_q(n,k,d)$	$\equiv n-g_q(k,d)+d-\lceil d/q^k \rceil$, p.24.
(n,s)-set in $GF(q)^r$	A set having n,r-tuples over GF(q) such that any s of them are linearly independent, p.24.
$\max_s(r,q)$	The maximal value of n for which an (n,s)-set in $GF(q)^r$ exists, p.24.
t[n,k]	The minimal covering radius of a binary [n,k] code, p.27.
$b_q(k,d)$	$\equiv n_q(k+1,d)-n_q(k,d)$, p.29.
b(k,d)	$\equiv n(k+1,d)-n(k,d)$, p.29.
$t_k(q)$	The maximum number of (-1)'s in any vector of $S_k(q)$, p.43.
wt(a+C)	= d(a,C), p.44.
$\mathcal{C}_f(u_1,u_2,..,u_p)$	The set of all k x n', $n' = \sum_{i=1}^{p} (2^{u_i}-1)$ matrice G' for which the distance between any two vectors of C'(generated by G') is at most $\sum_{i=1}^{p} 2^{u_i-1}$, p.50.

$\hat{U}(k,u)$	Set of all u-dimensional subspaces of $GF(2)^k$, p.50.
$U(k,u)$	$= \{\hat{U} \setminus \{0\} : \hat{U} \in \hat{U}(k,u)\}$, p.50.
$a \mid b$	a devides b

SYNOPSIS

OF THE

Ph. D DISSERTATION

ON OPTIMUM CODES AND THEIR COVERING RADII

BY

Madhu Sudan Garg
Department of Mathematics, Indian Institute of Technology,
Kanpur-208016 (India)

A linear code C of length n, dimension k and minimum distance d, over the Galois field GF(q) is called an [n,k,d] code. If q = 2 then the code is called a <u>binary code</u>. In 1965 Solomon and Stiffler [8] proved that for given k and d, the minimum value of n (denoted by $n_q(k,d)$) satisfies

$$n_q(k,d) \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \equiv g_q(k,d),$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. However the particular case q = 2 was first proved by Griesmer [5]. An $[n_q(k,d),k,d]$ code is called an <u>optimum code</u> and a $[g_q(k,d),k,d]$ code, if it exists is called a <u>Griesmer code</u>. For simplicity one usually writes $n(k,d)$ and $g(k,d)$ for $n_2(k,d)$ and $g_2(k,d)$ respectively.

Determining $n_q(k,d)$ for given k,d and q is a difficult task. In recent years many researchers like Baumert and McEliece [1], van Tilborg [10], Dodunekov and Manev [4] and

many others have tried to find $n(k,d)$ for $k \leq 8$. For $q > 2$ little is known about $n_q(k,d)$.

Another parameter of basic importance for a code is its covering radius, i.e., the weight of a maximum weight coset leader. Determining covering radius $R(C)$ for a given code C is another difficult task. A number of lower and upper bounds on $R(C)$ have been obtained by many researchers. For a survey of known results on the covering radius reader is referred to [2],

The present dissertation aims at determining (1) lower and upper bounds on the covering radius of an optimum code that are better than the other known bounds restricted to optimum codes (2) bounds on $n(9,d)$ and $n_4(k,d)$; $k = 3,4$. As an application of (1) bounds on the covering radius of a Simplex code over GF(q) (an open problem posed by Janwa [6]) and exact covering radius of many optimum codes, like MacDonald codes are obtained.

The covering radius of a binary Griesmer code C satisfies the inequality

$$R(C) \geq g(k,d) - 2^{k-1}.$$

Examples of binary Griesmer codes for which the above bound is better than the best lower bound $t[n,k]$ (= the minimum covering radius of any binary $[n,k]$ code) [2]; are given.

For given k,d and q , let the number '$n_q(k+1,d)-n_q(k,d)$' be denoted by $b_q(k,d)$. Obviously $b_q(k,d) \geq 1$. However if

$k \geq 2$ and $d \geq 5$ then it is shown that $b_2(k,d) \leq \lceil \frac{d}{2} \rceil - 1$. The following three results are proved.

Theorem 1 : The covering radius of any $[n,k,d]$ code C with $n < n_q(k+1,d)$ satisfies $R(C) \leq d-(n_q(k+1,d)-n)$.

Theorem 2 : The covering radius of an optimum code C satisfies $R(C) \leq d-b_q(k,d)$.

Theorem 3 : An $[n_q(k,d),k,d]$ code has covering radius $d-b_q(k,d)$ if and only if there exists an $[n_q(k+1,d),k+1,d]$ code with $b_q(k,d)$ equivalent coordinates.

The importance of Theorem 2 is demonstrated by using it to (1) determine exact covering radius of many optimum codes and (2) show that the optimum codes with covering radius d-1 are normal.

In [7] Janwa has shown that for an optimum code C, $R(C) \leq d-\lceil d/q^k \rceil$. The following theorem shows that the bound given by Theorem 2 is better than this bound.

Theorem 4 : If $n_q(k,d) = g_q(k,d)$ or $d \leq q^k$ then $d-b_q(k,d) \leq d-\lceil d/q^k \rceil$ However if $n_q(k,d) = g_q(k,d) + t$ and $n_q(k+1,d) = g_q(k+1,d) + t_1$, $0 \leq t < t_1$; then $d-b_q(k,d) < d-\lceil d/q^k \rceil$.

Let $\max_s(r,q)$ denote the maximum number of r-tuples over GF(q) with any s of them being linearly independent. If C is an $[n,n-r,d]$ code, then any d-1 columns of any parity check matrix for C are linearly independent. Hence $\max_{d-1}(r,q) \geq n$. For $n > r$ the converse is also true. Using this concept it

is shown that the bound given by Theorem 2 is the best possible for $b_q(k,d) = 1$. If $k_q[n,d] = \max\{ k$: there exists an $[n,k,d]$ code$\}$ then the following theorem explores relations among various parameters of a code.

Theorem 5 : Let $n_q(k,d) = n$ and let $n-k = r$. Then the following are equivalent

(i) $\max_{d-1}(r,q) = n$,

(ii) $k_q[n,d] = k_q[n+1,d] = k$,

(iii) $b_q(k,d) \geq 2$,

(iv) every $[n,k,d]$ code has covering radius $\leq d-2$.

Let $S_k(q)$ denotes the $[(q^k-1)/(q-1), k, q^{k-1}]$ Simplex code over GF(q). It is known that $R(S_k(2)) = 2^{k-1}-1$. If $q > 2$, then almost nothing is known about $R(S_k(q))$. In [6] Janwa has posed this as an open problem. Since $S_k(q)$ is an optimum code,it is maximal [3] and hence $R(S_k(q)) \leq q^{k-1}-1$ [2]. This bound can be further improved for certain values of k and q.

Theorem 6 : (i) If q is even then $R(S_2(q)) = q-1$.

(ii) If q is odd then $R(S_2(q)) \leq q-2$. This bound is shown to be attained for $q = 3$ and 5.

(iii) $R(S_k(q)) \leq q^{k-1}-2$, for $k \geq 3$ and $q = 3$ or 4. The bound is attained for $k = q = 3$.

Let $t_k(q)$ denotes the maximum number of (-1)'s in any vector of $S_k(q)$. A lower bound on $R(S_k(q)$ is given by following two theorems.

Theorem 7 : $R(S_k(q)) \geq n-t_k(q)$ and $t_m(q) = q^{m-k}\, t_k(q)$ for $m \geq k$.

Theorem 8 : (i) $R(S_k(3)) \geq \dfrac{2.3^{k-1} - 3^{k-2} - 1}{2}$

(ii) $R(S_k(4)) \geq \dfrac{3.4^{k-1} - 2.4^{k-2} - 1}{3}$

(iii) $R(S_k(5)) \geq \dfrac{3.5^{k-1} - 2.5^{k-2} - 1}{4}$

Tilborg [9] has shown that if $d > t2^{k-1}$, where $t = \lfloor (-3 + \sqrt{8k+1})/2 \rfloor$, then $n(k,d) = g(k,d)$. This bound is further improved. Let $s = \lceil d/2^{k-1} \rceil$ and let $v_1, v_2, .., v_t$ be positive integers defined inductively as follows.

v_1 is the largest integer $u < k$ for which $\sum_{i=1}^{\min\{u-2,s+1\}} (u-i) \leq sk$

Suppose $v_1, v_2, \ldots, v_{j-1}$ are defined for some $j < t$. Choose v_j to be the largest integer $u < v_{j-1}$ for which $\sum_{i=1}^{\min\{u-2,s-j+2\}} (u-i) \leq sk - \sum_{i=1}^{j-1} v_i$.

Finally let $v_t = sk - \sum_{i=1}^{t-1} v_i$.

Theorem 9 : $n(k,d) = g(k,d)$ for $d \geq (t-1)2^{k-1} - v$, where $v = 2^{v_1-1} + 2^{v_2-1} + \ldots + 2^{v_t-1} + 2^{v_t-2} + \ldots + 2$.

In particular if $k = 9$, then the above theorem yields that $n(9,d) = g(9,d)$ for $d \geq 337$. The equality is also shown to hold for $257 \leq d \leq 322$ by constructing certain 9-dimensional binary Griesmer codes. However the case $323 \leq d \leq 336$ remains unsolved.

A table of bounds on $n(9,d)$ for $d \leq 256$ is formed using known results. Bounds are further improved by constructing certain binary 9-dimensional codes and by showing nonexistence of certain binary linear codes. The results are summarized by following four theorems.

Theorem 10: $n(9,66) \leq 145$, $n(9,68) \leq 149$, $n(9,70) \leq 153$, $n(9,72) \leq 156$, $n(9,80) \leq 172$, $n(9,86) \leq 184$ and $n(9,88) \leq 187$.

Theorem 11: If $3 \leq i \leq k-4$ and $k \geq 9$ then $n(k,2^{k-i}) \geq g(k,2^{k-i})+3$.

Theorem 12: Binary $[53,9,24]$, $[101,10,48]$, $[116,9,56]$, $[125,10,60]$, $[196,9,96]$, $[228,9,112]$ and $[354,9,176]$ codes do not exist .

Theorem 13: Let $d = 2^{k-4}-2^m$, $k \geq 10$, $m \geq 0$. Then $n(k,d) \geq g(k,d) + 3$.

If $q > 2$ then much less is known about $n_q(k,d)$. Dodunekov [3] has generalized some of the results on binary codes to codes over GF(q). Using his results the following theorem is proved.

Theorem 14: $$n_4(3,d) = \begin{cases} g_4(3,d) & , \text{ if } d \neq 7,8 \\ g_4(3,d)+1 & , \text{ if } d = 7,8 \end{cases} .$$

Let $q = p^m$, $m = ts$ and let $n' = (p^m-1)/(p^s-1)$. If $\{\alpha_1,\alpha_2,..,\alpha_t\}$ is any basis for $GF(p^m)$ over $GF(p^s)$ and if $\beta_1,\beta_2,\ldots,\beta_t$ are the rows of a generator matrix for $S_t(p^s)$, then the linear mapping f from $GF(p^m)$ into $GF(p^s)^{n'}$ satisfying $f(\alpha_i) = \beta_i$ is an isomorphism. Moreover if C is any $[n,k,d]$ code over $GF(p^m)$ then mapping each codeword $(x_1,x_2,\ldots,x_n)$ to $(f(x_1),f(x_2),\ldots,f(x_n))$ one gets an $[n(p^m-1)/(p^s-1),tk,(p^s)^{t-1}d]$ code over $GF(p^s)$. This observation is used to prove the following results.

Theorem 15: If $k \geq 4$ and $d \in \{4^{k-2}-1, 4^{k-2}, 2.4^{k-2}-5,$

$2.4^{k-2}-4$, $3.4^{k-2}-5$, $3.4^{k-2}-4\}$ then $n_4(k,d) > g_4(k,d)$.

Theorem 16 : If $d \in \{3,4,7,8,15,16,43,44\}$ or $25 \leq d \leq 32$ then $n_4(4,d) > g_4(4,d)$.

Theorem 17 : $n_4(4,d) = g_4(4,d)$ for $45 \leq d \leq 64$, $93 \leq d \leq 96$, $105 \leq d \leq 128$, $141 \leq d \leq 144$, $157 \leq d \leq 160$, $173 \leq d \leq 192$, $201 \leq d \leq 208$ and $d \geq 213$.

Theorem 18 : $n_4(4,4d) \leq 5g_{16}(2,d)$.

REFERENCES

[1] L.D. Baumert and R.J. McEliece, ''A note on the Griesmer bound'', IEEE Trans. Inform. Theory, Vol. IT-19, pp. 134-135, 1973.

[2] G.D. Cohen, M.R. Karpovsky, H.F. Mattson Jr., and J.R. Schatz, ''Covering radius-Survey and recent results'', IEEE Trans. Inform. Theory, Vol. IT-31, pp. 328-343, 1985.

[3] S.M. Dodunekov, ''Minimal block length of a linear q-any code with specified dimension and code distance'', Prob. Inform. Transm., Vol. 20, pp. 239-249, 1984.

[4] S.M. Dodunekov and N.L. Manev, ''An improvement of the Griesmer bound for some small minimum distances'', Discrete Applied Mathematics, Vol. 12, pp. 103-114, 1985.

[5] J.H. Griesmer, ''A bound for error-correcting codes'', IBMJ. Res. Develop. Vol. 4, pp. 532-542, 1960.

[6] H. Janwa, ''Some new upper bounds on the covering radius of binary linear codes'', IEEE Trans. Inform. Theory, Vol. IT-35, pp. 110-122, 1989.

[7] H. Janwa, ''On the covering radius of q-any codes'', to appear.

[8] G. Solomon and J.J. Stiffler, ''Algebraically punctured cyclic codes'', Information and control, Vol.8, pp. 170-179, 1965.

[9] H.C.A. van Tilborg, ''On the uniqueness resp. nonexistence of certain codes meeting the Griesmer bound'', Information and control, Vol. 44, pp. 16-35, 1980.

[10] H.C.A. van Tilborg, ''The smallest length of binary 7-dimensional linear codes with prescribed minimum distance'', Discrete Mathematics, Vol. 33, pp. 197-207, 1981.

CHAPTER I

INTRODUCTION

An [n,k,d] code is a linear code over the Galois field GF(q) with q elements ($q = p^m$, for some prime p and for some positive integer m) of length n, dimension k and minimum distance d. For given k and d the minimal value of n for which an [n,k,d] code exists is denoted by $n_q(k,d)$ and an $[n_q(k,d), k, d]$ code is called an optimum code. In 1965 Solomon and Stiffler [41] showed that

$$n_q(k,d) \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \equiv g_q(k,d).$$

However the particular case q = 2 was first proved by Griesmer [22]. The bound $g_q(k,d)$ is usually called the Griesmer bound and a $[g_q(k,d), k, d]$ code, if it exists is called a Griesmer code.

Determining $n_q(k,d)$ for given k,d and q is a difficult task. In recent years many researchers like Baumert and McEliece [2], van Tilborg [43], Dodunekov and Manev [16] and many others have tried to find $n_2(k,d)$ for $k \leq 8$. For $q > 2$ little is known about $n_q(k,d)$ [12].

Another parameter of basic importance for a code is its covering radius, i.e., the weight of a maximum weight coset leader. Determining covering radius R(C) for a given code C

is another difficult task. A number of lower and upper bounds on R(C) have been obtained by Cohen, Karpovsky, Mattson and Schatz [8], Delsarte [11], Helleseth, Kløve and Mykkeltveit [25], Janwa [32] and by many others.

The present dissertation aims at determining (i) lower and upper bounds on the covering radius of an optimum code that are better than the other known bounds restricted to optimum codes (ii) bounds on $n(9,d)$ and $n_4(k,d)$; $k = 3,4$. As an application of (i) bounds on the covering radius of a Simplex code over GF(q) (an open problem posed by Janwa [32]) and exact covering radius of many optimum codes are obtained. The organization of the dissertation is as follows.

Chapter II contains some essential known results on properties of optimum codes, bounds on $n_q(k,d)$, residual of a code, covering radius and normality. Some easy consequences of the concept of a residual code like (i) determination of an upper bound on the minimum distance of certain cyclic codes and (ii) characterization of binary Simplex code and Reed-Muller code of first order are also derived.

Lower and upper bounds on the covering radius of an optimum code are obtained in Chapter III. The bounds are shown to be better than other known bounds restricted to optimum codes. Exact covering radius of a subcode of index q of an optimum code is found and used to give a sufficient

condition for the normality of optimum codes. Theorems giving relations among different parameters of a code like $n_q(k,d)$, covering radius, $\max_s(r,q)$ (the maximum number of r-tuples over GF(q) such that any s of them are linearly independent), $d_q[n,k]$ (the maximum minimum distance of any $[n,k]$ code) and $k_q[n,d]$ (the maximum dimension of any code of length n and minimum distance d) are also established. Finally the results are used to determine bounds on the covering radius of a Simplex code over GF(q). This was posed as an open problem by Janwa [32].

In 1980 Tilborg [42] proved that for a fixed k if $d > \lceil(-3 + \sqrt{8k+1})/2\rceil 2^{k-1}$ then $n_2(k,d) = g_2(k,d)$. This bound is improved in Chapter IV which in particular shows that $n_2(9,d) = g_2(9,d)$ for $d \geq 337$. A table of bounds on $n_2(9,d)$ for $d \leq 256$, using known results is also constructed. The bounds are further improved by showing nonexistence of certain codes of dimension 9 and by constructing certain 9-dimensional binary codes. Some general results on the lower bound for $n_2(k,d)$ are also derived.

Chapter V is devoted to the problem of determining $n_4(k,d)$ for k=3 and 4. Basic results of Dodunekov [12] on $n_q(k,d)$ are used to determine $n_4(3,d)$ and lower bounds on $n_4(4,d)$. An isomorphism between codes over $GF(p^m)$ and codes over $GF(p^s)$,

is a positive divisor of m, is established. This relation to show nonexistence of certain $[g_4(k,d), k, d]$ codes over and for determining an upper bound for $n_4(4,d)$.

CHAPTER II

PRELIMINARIES AND SURVEY

A field is a nonempty set F with two binary operations denoted by '+' and '.' satisfying (i) (F,+) is an abelian group (ii) nonzero elements of F form an abelian group under '.' and (iii) $a.(b+c) = a.b + a.c$ for all $a,b,c \in F$. If F has finite number of elements, then F is called a finite field. Galois has shown that a finite field with q elements exists if and only if $q = p^m$, for some prime p and for some positive integer m. Moreover a finite field with q elements is unique up to isomorphism and is usually denoted by GF(q). For results on finite fields we refer to [38]. Let $GF(q)^n$ denotes the set of all n-tuples over GF(q). For $x,y \in GF(q)^n$, weight of x (denoted by wt(x)) is the number of nonzero components in x and distance between x and y (denoted by d(x,y)) is the weight of x-y. For a subset S of $GF(q)^n$, distance between x and S (denoted by d(x,S)) is $\min \{d(x,y): y \in S\}$. If $x = (x_1,x_2,..,x_n)$ and $y = (y_1,y_2,..,y_n)$ then x_i is called the ith coordinate of x and the vector $(x_1y_1,x_2y_2,..,x_ny_n)$ (denoted by x*y) is called the intersection of x and y.

A linear code C of length n is a subspace of $GF(q)^n$. If C has dimension k and minimum distance d then it is called an [n,k,d] code. Elements of a code are called codewords and the phrase ''the ith coordinate of C ''means the i-th

coordinate function of C . If q = 2 then the code is called a binary [n,k,d] code. Any [n,k] linear code C, being a subspace can be described either by giving a basis for it or by identifying an (n-k) x n matrix H whose solution space is C. The matrix H is called a parity check matrix for C. A kxn matrix whose rows form a basis for C is called a generator matrix for the [n,k] code C.

Let $G_k(q)$ be the matrix whose columns consist of all nonzero k-tuples over GF(q) with any two columns linearly independent. Then the solution space of $G_k(q)X = 0$ is a $[(q^k-1)/(q-1),((q^k-1)/(q-1))-k, 3]$ code (denoted by $H_k(q)$). $H_k(q)$ is called the Hamming code. On the other hand the code generated by the matrix $G_k(q)$ is a $[(q^k-1)/(q-1),k,q^{k-1}]$ code (denoted by $S_k(q)$). $S_k(q)$ is called the Simplex code. It is easy to see by induction that the matrix

$$G_{k+1}(q) = \left[\begin{array}{c|c|c|c|c} 0\ 0\ \ldots 0 & \begin{matrix}1\\0\\\vdots\\0\end{matrix} & 1\ 1\ \ldots 1 & \ldots & \alpha_{q-2}\ \alpha_{q-2}\cdots\alpha_{q-2} \\ G_k(q) & & G_k(q) & \ldots & G_k(q) \end{array}\right]$$

is a generator matrix for $S_{k+1}(q)$, where $GF(q) = \{0,1,\alpha_1,\alpha_2,\ldots,\alpha_{q-2}\}$ Moreover every nonzero codeword of $S_{k+1}(q)$ has weight q^k . The binary Simplex code of dimension k is denoted by S_k.

The code generated by the matrix

$$\left[\begin{array}{c|c} 1\ 1\ \ldots\ 1 & 1 \\ \hline G_k(2) & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \end{array}\right]$$

is called the first order binary Reed-Muller code (denoted by RM(1,k)). Note that RM(1,k) is a $[2^k,\ k+1,\ 2^{k-1}]$ code.

Let M be a matrix written as the juxtaposition of matrices A and B , that is $M = [A|B]$. A is denoted by $[M \setminus B]$. For $1 \leq u \leq k-1$, the code $C_{k;u}(q)$ generated by the matrix

$$\left[G_k(q) \setminus \frac{0}{G_u(q)}\right] \tag{2.1}$$

is called a MacDonald code [16] . Note that $C_{k;u}(q)$ is a $[(q^k-q^u)/(q-1),\ k,\ q^{k-1} - q^{u-1}]$ code [12]. We usually write $C_{k;u}$ for $C_{k;u}(2)$.

Two codes are said to be equivalent if a generator matrix for one can be obtained from a generator matrix for the other by a column permutation followed by multiplying some columns by nonzero elements of GF(q) . Every $[(q^k-q^u)/(q-1),\ k\ ,\ q^{k-1} - q^{u-1}]$ code is equivalent to $C_{k;u}(q)$ [12].

If $x = (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n) \in GF(q)^n$ then $x.y = \sum_{i=1}^{n} x_i y_i \pmod q$ defines an inner product on

$GF(q)^n$. If C is an [n,k] code then $C^{\perp} = \{ u \in GF(q)^n : u.c = o,$ for all $c \in C \}$, called dual of C, is an [n,n-k] code. Moreover a generator matrix for C is a parity check matrix for $C^{\perp}$ and vice versa. Thus the Simplex code $S_k(q)$ is the dual of the Hamming code $H_k(q)$. A code C is called self orthogonal or weakly self dual if $C \subseteq C^{\perp}$. If $C = C^{\perp}$ then C is called self dual.

Let A_i, $o \leq i \leq n$; denote the number of codewords of weight i in an [n,k,d] code C. Then $\langle A_i \rangle$, $o \leq i \leq n$; is called the weight distribution of C. Obviously $A_o = 1$ and $A_i = o$ for $1 \leq i \leq d-1$. In 1963 MacWilliams determined a set of equations, usually called 'MacWilliams equations' relating the weight distribution of $C^{\perp}$ to the weight distribution of C [38].

Proposition 2.1 [38]. Let $\langle A_i \rangle$ and $\langle B_i \rangle$, $o \leq i \leq n$; be the weight distributions of a linear code C and its dual $C^{\perp}$ respectively. Then

$$B_m = |C|^{-1} \sum_{i=o}^{n} A_i \; K_m(i) \;, \quad o \leq m \leq n \;;$$

where $K_m(x) = \sum_{j=o}^{m} (-1)^j \binom{n-x}{m-j} \binom{x}{j} = (-1)^m K_m(n-x), \; o \leq m \leq n \;;$

$K_m(x)$ is called the Krawtchouk polynomial.

Determining $K_m(x)$ is a laborious task. The help of a computer is taken to compute $K_m(x)$ for certain values of m,n and x

For given k and d the minimum value of n for which there exists an [n,k,d] code is denoted by $n_q(k,d)$. In 1965 Solomon and Stiffler [41] have shown that

$$(2.2) \qquad n_q(k,d) \geq \sum_{i=0}^{k-1} \lceil d/q^i \rceil \equiv g_q(k,d) ,$$

where $\lceil x \rceil$ denotes the least integer $\geq x$. However the particular case q = 2 was first proved in 1960 by Griesmer [22]. For simplicity one usually writes n(k,d) and g(k,d) for $n_2(k,d)$ and $g_2(k,d)$ respectively. The lower bound on $n_q(k,d)$ given by (2.2) is usually called the Griesmer bound and the code $[g_q(k,d), k, d]$, if it exists is called a code meeting the Griesmer bound or simply a Griesmer code. It is easy to verify that Hamming, Simplex and MacDonald codes are examples of Griesmer codes. An $[n_q(k,d), k, d]$ code is called an optimum code.

If C is any $[(q^k-1)/(q-1), k, q^{k-1}]$ code , then C is a Griesmer code and hence no coordinate of it is identically zero. Moreover if any two coordinates of C are equivalent then on applying row and column operations, if necessary any generator matrix for C can be put in the form

$$\left[\begin{array}{cc|ccc} 1 & 1 & * & * \ \ldots & * \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & G' & \\ 0 & 0 & & & \end{array}\right] ,$$ where *'s are some elements in GF(q).

Clearly G' generates $[((q^k-1)/(q-1))-2,\ k-1,\ q^{k-1}]$ code. But such a code does not exist as $g_q(k-1,q^{k-1}) = ((q^k-1)/(q-1))-1$. So coordinates of C are inequivalent. This gives an independent proof of the following result of MacDonald (1960).

Theorem 2.1. Any $[(q^k-1)/(q-1),\ k,\ q^{k-1}]$ code is equivalent to $S_k(q)$.

II. A. OPTIMUM CODES

Let $s = \left\lceil \frac{d}{2^{k-1}} \right\rceil$ and let $s.2^{k-1} - d = \sum_{i=1}^{p} 2^{u_i-1}$,

where $k > u_1 > u_2 > \ldots > u_p > o$. Let G_k be a generator matrix for S_k and let U_i be a $k \times (2^{u_i}-1)$ submatrix of G_k of rank u_i, $1 \leq i \leq p$. If

$$(2.3) \qquad \sum_{i=1}^{p} u_i \leq sk ,$$

then Solomon and Stiffler [41] have shown that the matrix

$$(2.4) \qquad [(\underbrace{G_k|G_k|\ldots|G_k}_{\text{s-copies}}) \setminus (U_1|U_2|\ldots|U_p)]$$

generates a binary $[g(k,d),\ k,\ d]$ code. Further if

$$(2.5) \qquad u_{i+1} = u_i - 1 \text{ for } i = s, s+1, \ldots, p-1,\ u_p \in \{1,2\}$$

and G is the matrix defined by

$$(2.6) \qquad G = [(\underbrace{G_k|G_k|\ldots|G_k}_{\text{s-copies}}) \setminus (U_1|U_2|\ldots|U_t|U \quad T|R)],$$

where (i) U is a $kx(2^u-1)$ submatrix of G_k of rank u ;

(ii) T is a $kx(u+1)$ submatrix of U of rank u, whose columns add upto the zero vector ;

(iii) R as a set contains a nonzero vector, if d is odd and null set otherwise;

(iv) no column occurs more than s-times among the deleted columns ;

then G generates a binary [g(k,d), k,d] code [3]. The most general result of this type is that of Belov [3].

Proposition 2.2 [3]. For given k and d let s and u_i's be as defined above. If $\sum_{i=1}^{\min\{p,s+1\}} u_i \leq sk$ or $u_{i+1} = u_i - 1$ for $i = s,s+1,\ldots,p-1$ and $u_p \in \{1,2\}$ then $n(k,d) = g(k,d)$.

Belov [3] conjectured that for s = 1 conditions (2.3) and (2.5) are also necessary for the existence of a binary [g(k,d), k, d] code. His conjecture was proved in parts by Logacev [36] , van Tilborg [42] and Helleseth [23] by showing that the only [g(k,d), k, d] ; $d \leq 2^{k-1}$ codes that exist come from the constructions given by (2.4) or (2.6). In particular the result of Helleseth gives

Proposition 2.3 [16]. For $2^{k-1}-2^{k-i}+3 \leq d \leq 2^{k-1}-2^{k-i-1}-2^{i}$, $1 \leq i \leq \lfloor (k-2)/2 \rfloor$, the inequality 'n(k,d) > g(k,d)' holds and for the other values of $d \leq 2^{k-1}$, n(k,d) = g(k,d).

As a consequence of Proposition 2.2 Tilborg [42] derived that for a fixed k if

$$d > \lceil (-3 + \sqrt{8k+1})/2 \rceil \, 2^{k-1}, \tag{2.7}$$

then n(k,d) = g(k,d).

Further for $s > 1$ Helleseth and van Tilborg [26] have constructed a new class of codes meeting the Griesmer bound for $2^{k-1}+2^{k-3}-15 \leq d \leq 2^{k-1} + 2^{k-3} - 8$, $k \geq 7$. Thus above stated conjecture of Belov does not hold for $s > 1$. In [24] Helleseth gave a new description of binary Griesmer codes that includes the earlier constructions given by Solomon and Stiffler [41], Belov, Logachev and Sandimirov [4] and Helleseth and van Tilborg [26]. He also gave several new families of binary Griesmer codes that are not attainable by any of the previously known constructions.

If C is a $[g_q(k,d)+t, k, d]$, $t = 0$ or 1 ; code then Dodunekov has shown that there exists a generator matrix for C in which each row has weight d or d+t [12]. However for binary codes Dodunekov and Manev have proved the following general result

Proposition 2.4 [16]. If C is a binary $[n,k,d]$ code with $n = g(k,d)+t$, for some nonnegative integer t, then C has a generator matrix with each row having weight between d and d+t.

The case $t = o$ of Proposition 2.4 was first proved by Tilborg [42]. An easy consequence of Proposition 2.4 is the following

Proposition 2.5 [16]. If C is a binary $[g(k,d)+1, k, d]$ code with $d = 2^m t$, $3 \leq t < 2^{k-m-2}-1$, then $wt(c) \equiv o \pmod{2^m}$ for every $c \in C$.

Following proposition summarizes the properties of a binary Griesmer code deduced by Busschbach, Gerretzen and van Tilborg [7], Dodunekov and Manev [15] and van Tilborg [12].

Proposition 2.6. Let C be [g(k,d), k, d] code. Then any generator matrix for C has no column repeated more than $\lceil d/2^{k-1} \rceil$ times and some column repeated exactly $\lceil d/2^{k-1} \rceil$ times. Moreover if $d = 2^m.t$, then $wt(c) \equiv o \pmod{2^m}$ for all $c \in C$.

Determining $n_q(k,d)$ for given k and d in general is a difficult task. In recent years Baumert and McEliece [2], van Tilborg [43], Dodunekov and Manev [16] and many others have worked on determining n(k,d) for small values of k. If $k \leq 4$ and d any positive integer then by Proposition 2.2 $n(k,d) = g(k,d)$. However this is not the case for $k \geq 5$. Baumert and McEliece [2] have determined the value of n(k,d) for k = 5 and 6. In fact they proved the following

Proposition 2.7 [2]. $n(k,d) = g(k,d)+1$ for $k = 5$, $3 \leq d \leq 6$ or $k = 6$, $3 \leq d \leq 14$ or $19 \leq d \leq 20$. For all other values of d with k = 5 or 6, $n(k,d) = g(k,d)$.

van Tilborg [43] calculated n(7,d) for all values of d and gave several values of d for which $n(7,d) = g(7,d) + 2$. Some classes of binary codes satisfying the inequality '$n(k,d) \geq g(k,d) + 2$' were determined by Dodunekov and

Manev [16]. Their results are summarized by following two propositions.

Proposition 2.8 [16]. If $d \leq 2^k$ and $n(k,d) \geq g(k,d)+t$, then $n(k+1,d) \geq g(k+1,d)+t$.

Proposition 2.9 [16]. $n(k,d) \geq g(k,d) +2$ for the following values of k and d

(i) $d = 2^{k-i}, 2^{k-i}-2$; $3 \leq i \leq k-4$;

(ii) $d = 3.2^{k-i}, 3.2^{k-i} - 2$; $4 \leq i \leq k-4$;

(iii) $d = 5.2^{k-i}(k \geq 9), 5.2^{k-i}-2(k \geq 10)$; $5 \leq i \leq k-2$;

(iv) $d = 7.2^{k-i}(k \geq 7), 7.2^{k-i}-2\ (k \geq 8)$; $5 \leq i \leq k-1$.

They also determined bounds on $n(8,d)$, which were further improved by Dodunekov, Helleseth, Manev and Ytrehus [14],[17]. In [27] Helleseth and Ytrehus showed the nonexistence of some binary 8-dimensional codes by first proving the following two propositions.

Proposition 2.10 [27]. Let C be a binary $[n,k,d]$ code and let $\{B_i\}$, $o \leq i \leq n$; be the weight distribution of $C^{\perp}$. If $B_i \neq o$ then an $[n-i, k-i+1, d]$ code exists.

Proposition 2.11 [27]. Let C be a binary code and let $c_1, c_2 \in C$. If $c_1 = (1,1,\ldots,1,o,o,\ldots,o)$ and $c_2 = (c_2^1 | c_2^o)$, c_2^o being part of c_2 below the zero entries of c_1, then $wt(c_1+c_2) = wt(c_1) - wt(c_2) + 2wt(c_2^o)$.

The updated detailed information of bounds on $n(8,d)$

can be obtained from the table of Verhoeff [44] on $d_2[n,k]$, $1 \leq k \leq n \leq 127$.

Note that for $k \leq 8$ there is no $d > 2^{k-1}$ for which $n(k,d) > g(k,d)$. However this is not true for $k \geq 9$ as Logachev [37] has proved that a binary $[g(9,336),9,336]$ code does not exist.

Dodunekov [12] generalized (2.3) and (2.5) to codes over GF(q) and gave a set of sufficient conditions on k, d and q for which $n_q(k,d) = g_q(k,d)$. The following three propositions summarize his results.

Proposition 2.12 [12]. Let $s = \lceil d/(q-1)q^{k-1} \rceil$ and let $s(q-1)q^{k-1} - d = \sum_{i=o}^{p} a_i q^{u_i-1}$, $k = u_o > u_1 > \ldots > u_p > o$, $o \leq a_o \leq q-2$, $1 \leq a_i \leq q-1$ for $1 \leq i \leq p$. The optimum code meets the Griesmer bound if any one of the following holds.

(2.8) $a_o = o$ and $\sum_{i=1}^{p} u_i \leq sk$;

(2.9) $a_o = o$, $u_{i+1} = u_i - 1$, for $i = s, s+1, .., p-1$, and $n_q(u_s+1, d_1) = g_q(u_s+1, d_1)$ for $d_1 = q^{u_p-1} + \sum_{i=s}^{p} (q-1-a_i) q^{u_i-1}$;

(2.10) $a_o > o$ and $s(q-1) \geq \sum_{i=o}^{p} a_i$;

(2.11) $d > [(k-2)(q-1) - (q-2)] q^{k-1} - 2q$, $k \geq 3$;

(2.12) $q = 2^m$, $k = 3$ and $d \leq q+2$;

(2.13) q odd, $k = 3$ and $d \leq q+1$, except for $d=q$.

Proposition 2.13 [12]. If $k \geq q \geq 3$ and $2q^i < d \leq q^{i+1}$ for $0 \leq i \leq k-q$, then $n_q(k,d) > g_q(k,d)$.

Proposition 2.14 [12]. If $q|d$ and $n_q(k,d) = g_q(k,d)$ then $n_q(k,d-a) = g_q(k,d-a)$, for all $1 \leq a \leq q-1$. Conversely if $n_q(k,d-a) > g_q(k,d-a)$, for some $1 \leq a \leq q-1$ then $n_q(k,d-b) > g_q(k,d-b)$, for all $0 \leq b \leq a$.

II.B. RESIDUAL CODE

For showing nonexistence of certain codes Tilborg introduced binary residual codes [43]. Dodunekov further extended it for codes over GF(q) [12] . Let C be an $[n,k,d]$ code with generator matrix G and let c be the first row of G. Then the residual code of C with respect to c (denoted by res(C,c)) is the code generated by the restriction of G to those columns where c has a zero entry. If only weight w of c is relevant we usually write res(C,w) for res(C,c). For $w < d + \lceil w/q \rceil$, res(C,w) is an $[n-w, k-1, d_o]$, $d_o \geq d-w + \lceil w/q \rceil$ code [12].

The concept of a residual code is useful in many respects. For example :

1. To show the nonexistence of certain codes : Nonexistence of a binary $[n,k,d]$ code with even d implies nonexistence

of any binary [n+2d, k+1, 2d] or [n+2d-3, k+1, 2d-2] code.

2. To find an upper bound for the minimum distance of certain binary cyclic codes : We need the following proposition.

Proposition 2.15 [38 ; p.203]. If C is a binary [n,k,d] cyclic code with $\delta-1$ consecutive zeros then $d \geq \delta$.

Example 2.1 Let C be the [9,7,d] binary cyclic code generated by $g(x) = x^2+x+1$. Then $\delta = 2$ and so $2 \leq d \leq 3$. Res(C,3) is a $[6,6,d_0]$ code. By the Griesmer bound $1 \geq d_0 \geq d - \lfloor 3/2 \rfloor$; hence d = 2.

Example 2.2 Let C be the [15,9,d] binary cyclic code generated by $g(x) = x^6+x^4+x^3+x^2+1$. Then $\delta = 3$ and so $3 \leq d \leq 5$. Res(C,5) is a $[10,8,d_0]$ code, where $2 \geq d_0 \geq d-\lfloor 5/2 \rfloor$, $d \leq 4$ (the actual value).

Example 2.3 Let C be the [21,12,d] binary cyclic code generated by $g(x) = x^9 + x^8 + x^5 + x^4 + x^2 + x + 1$. Then $\delta = 5$ and so $5 \leq d \leq 7$. Now res(C,7) is a $[14,11,d_0]$ code with $2 \geq d_0 \geq d-\lfloor 7/2 \rfloor$; hence d = 5.

3. For showing uniqueness of certain binary codes with given weight distribution.

First part of the following theorem has been proved by J.E. MacDonald (IBM J. Res. Dev., 1960). However a simple proof is given for completeness.

Theorem 2.2 Let C be a binary $[n = n(k,d), k, d]$ code. (i) If all nonzero vectors in C have weight d then for some positive integer t, C is isomorphic to t. S_k(t-copies of the Simplex code S_k). (ii) If C has all one vector (denoted by $\underline{1}$) and every other nonzero vector has weight d then C is isomorphic to t-copies of the first order Reed-Muller code of dimension k (denoted by t.RM(1,k-1)).

Proof. (i) d is even. For if d = 2m+1, then res(C,d) is an $[n-d, k-1, d_o]$, $d_o \geq m+1$ code. Let $c_1, c_2 \in C$. Without loss of generality we can assume that $c_1 = (1\ 1\ \ldots 1\ o\ o \ldots o)$ ($\leftarrow n-d \rightarrow$) and let $c_2 = (c_2^1 | c_2^o)$, where c_2^1 is the first part of c_2 of length d and c_2^o is the remaining part of length n-d. Since $c_2^o \in$ res(C,d), $wt(c_2^o) \geq m+1$ and hence $wt(c_2^1) \leq m$ (as $wt(c_2) = d$). Therefore $wt(c_1+c_2) = d - wt(c_2^1) + wt(c_2^o) \geq 2m+2$, a contradiction. Let d = 2m . Then res(C,d) is an $[n-d, k-1, m]$ code with all nonzero vectors having weight m and hence m is even. Proceeding this way we have $d = 2^{k-1}t$ for some positive integer t. From Proposition 2.2 it is easy to see that $n(k,d) = g(k,d) = t(2^k-1)$. No coordinate of C could be zero and every coordinate repeats at most t times as $\lceil d/2^{k-1} \rceil = t$. Further no coordinate of C repeats fewer than t times for otherwise n will be less than $t(2^k-1)$. Hence C is isomorphic to $t.S_k$.

(ii) Let G be a generator matrix for C with first row of weight n and let G' be the matrix G with first row deleted.

The code C' generated by G' is a single weight code. Permuting columns of G' it may be written in the form

$$[\, G_{k-1} | G_{k-1} | \ldots | G_{k-1} | 0],$$

where G_k denotes a generator matrix of S_k. Thus $d = t2^{k-2}$. MacWilliams equation for B_1 gives $n+(n-t2^{k-1})(2^k-2)+(n-2n) = 0$ or $n = t2^{k-1}$. Hence by definition of a first order Reed-Muller code , C is isomorphic to t.RM(1,k-1).

II.C. THE COVERING RADIUS PROBLEM

Let C be an [n,k,d] code. Then spheres of radius [d-1/2] around codewords are disjoint, but they need not cover the whole space $GF(q)^n$. The smallest integer ρ , such that spheres of radius ρ around codewords cover the whole space is called the covering radius of the code. We will denote the covering radius of a code C by R(C) . A code C is called perfect if $R(C) = \lfloor (d-1)/2 \rfloor$ and quasiperfect if $R(C) = \lfloor (d-1)/2 \rfloor + 1$. Many equivalent statements for the covering radius are known [8] . For example the covering radius of a code is

(i) the weight of a maximum weight coset leader ;

(ii)the least integer ρ such that any (n-k)-tuple over GF(q) (called Syndrome) is a linear combination of some ρ or fewer columns of any parity check matrix of the code.

Determining covering radius of a given code in

general is a difficult task. In recent years much work has been done on finding bounds for the covering radius of a code. For a survey of results on covering radius reader is referred to [8], [32]. Some of the results that are needed for our refrence, are listed below for completeness.

If C is an [n,k,d] code and ρ is the least positive integer for which

$$\sum_{i=0}^{\rho} \binom{n}{i} (q-1)^i \geq q^{n-k} , \tag{2.14}$$

then $R(C) \geq \rho$.

Proposition 2.16 [8]. Appending an overall parity check or the zero parity check to the generator matrix of a given code increases the covering radius by one ; while puncturing a code on p coordinates reduces the covering radius by atmost p.

Let C_1 and C_2 be codes with dimensions k_1 and k_2 ($k_1 \geq k_2$) and generator matrices G_1 and G_2 respectively. A code C is said to be a <u>catenation</u> of C_1 and C_2 if C has a generator matrix $[G_1|G_2']$, where G_2' is G_2 with k_1-k_2 rows of zeros attached. The following proposition gives a lower bound on R(C) in terms of $R(C_1)$ and $R(C_2)$.

Proposition 2.17 [8]. If C is a catenation of two codes C_1 and C_2 then $R(C) \geq R(C_1) + R(C_2)$.

If a binary [n,k] code C has a generator matrix in the form

$$G = \left[\begin{array}{c|c} G_o & A \\ \hline O & G_1 \end{array}\right]$$

and if the codes generated by G_o and G_1 are C_o and C_1 respectively then Mattson [39] has proved the following relation between their covering radii.

Proposition 2.18 [39]. Let C, C_o and C_1 be as defined above then $R(C) \leq R(C_o) + R(C_1)$.

A lower bound for the covering radius of an [n,k] code is $t_q[n,k]$, the minimal covering radius of any [n,k] code. This was first defined by Cohen, Karpovsky, Mattson and Schatz [8] for q = 2. Obviously $t_q[n,k] \geq \rho$, where ρ is given by (2.14). In [21] Graham and Sloane have given a table of bounds on $t_2[n,k]$ for $1 \leq k \leq n \leq 64$.

A code C_1 is called a subcode (supercode) of C if $C_1 \subseteq C (C_1 \supseteq C)$. Moreover C_1 is said to be a subcode of **codimension** r of C, if there exist r distinct elements $a_1, a_2, .., a_r \in C$, such that $C = \bigcup_{i=1}^{r} (a_i + C_1)$, $(a_i + C_1) \cap (a_j + C_1) = \Phi$, $i \neq j$. An [n,k,d] code C is called maximal if there does not exist a proper supercode of C with same n and d. A code C is maximal if and only if $R(C) \leq d-1$ [8]. Dodunekov [12] has shown that any $[n_q(k,d),k,d]$ code C is maximal and hence

(2.15) $$R(C) \leq d-1.$$

Many upper bounds on the covering radius of a binary code C are known [8]. The best known upper bound on the covering radius of an [n,k,d] code C is given by Janwa [34].

Proposition 2.19 [34]. Let C be an [n,k,d] code, then

$$R(C) \leq n - \sum_{i=1}^{k} [d/q^i] = n - g_q(k,d) + d - [d/q^k] \equiv H_q(n,k,d).$$

If $n = g_q(k,d)$, then the above proposition immediately gives

$$R(C) \leq d - \lceil \frac{d}{q^k} \rceil. \tag{2.16}$$

The above inequality for q = 2 was first proved by Busschbach, Gerretzen and van Tilborg [7]. In [34] Janwa has extended the truth of (2.16) to any optimum code. Thus

Proposition 2.20 [34]. If C is an $[n_q(k,d),k,d]$ code then $R(C) \leq d - [d/q^k]$.

II.D. NORMALITY OF CODES

Graham and Sloane [21], while looking for binary [n,k] codes with smaller covering radius, defined the concept of normality. This was generalized to nonbinary linear codes by Janwa [32].

Let C be an [n,k] code with none of the coordinate identically zero. Fix $i \in \{1,2,\ldots,n\}$. For $\alpha \in GF(q)$, let

$$C_\alpha^{(i)} = \{ x \in C: \text{ ith coordinate of } x \text{ is } \alpha \} \text{ and let}$$

$$N^{(i)} = \max \{ \sum_{\alpha \in GF(q)} d(x, C_\alpha^{(i)}) : x \in GF(q)^n \}.$$

Then $N^{(i)}$ is called the norm of C with respect to the coordinate position i. Let $N = \min_{1 \leq i \leq n} N^{(i)}$. N is called the <u>norm</u> of C. The code is called <u>normal</u> if $N \leq qR(C) + (q-1)$.

With the above definition of the norm Theorems 6,11 and 12 of Graham and Sloane [21] are false (M.C. Bhandari and M.S. Garg, IEEE Trans. Inform Theory, pp. 953-954, 1990).

However with a modified definition of norm given by Cohen,)bstein and Sloane [9] these results are true. In the modified :finition $N^{(i)}$ is defined as above and N is said to be a norm : the code if $N \geq N^{(i)}$ for at least one coordinate i. This :finition is puzzling as the authors of [9] at one place say 'We deliberately do not insist that equality holds'' and on the :her place they say ''Of course we always choose N as small as)ssible''. Therefore the previous definition of the norm is better ; one obtains the unique value of the norm for a given code.

The hypothesis of Lemma 27 in [21] proved by Graham and Sloane :eds to be modified (M.C. Bhandari and M.S. Garg, IEEE Trans. Inform. ieory, pp. 653-654, 1990). Let F_i denotes the set of all binary -tuples except $\underline{0}$ and $\underline{1}$ (the all zero and all one vector respectively) id let $A_i \subseteq F_i$, $A_i^c = F_i \setminus A_i$ $(i = 1,2)$. Then the above lemma is :ated as follows.

:oposition 2.21 [21]. If A_1 and A_2 satisfy

$$(A_i+A_i) \cup (A_i^c+A_i^c) \supset A_i^c \quad (i = 1,2),$$

$$A_i+A_i^c \supset A_i \quad (i = 1,2),$$

$$\underline{1}+A_1 = A_1,$$

$$\underline{1} \in A_2+A_2^c$$

ien the code with parity check matrix

$$\left[\begin{array}{c|c|c|c} A_1 & A_1^c & 0 & 1 \\ \hline 0 & 1 & A_2 & A_2^c \end{array}\right]$$

as covering radius 2.

If we take $m_1 = 3$, $m_2 = 4$; $A_1 = \{$ olo,ool,lol,llo $\}$ and $_2 = \{$ llol,lllo,lloo,olll,olol,ollo,oloo $\}$, then A_1 and A_2 satisfy all

he conditions of Proposition 2.20 but the vector $(0001111)^{tr}$ can not be ritten as a sum of two columns of the parity check matrix given in roposition 2.20.

However Proposition 2.20 is true if A_1 and A_2 in addition satisfy εA_2+A_2.

In a recent paper Hou (IEEE Trans. Inform. Theory, pp. 683-685,1990) as proved the following upper bounds on the norm N of a code C

(i) $N \leq 2R(C) + [d/2] - 1$, $(d \geq 3)$ (ii) $N \leq 3R(C)-2 (R \geq 3)$.

The following theorem summarizes known sufficient conditions or a code to be normal [9],[31],[33].

roposition 2.22. Let C be an [n,k,d] code and let $H(C) = n_q(k,d) + d - \lceil d/q^k \rceil$. C is normal if any one of the following holds

i) $q = 2$ and $R(C_o^{(i)}) \leq R(C) + 2$, for some i ;

ii) $q = 2$, $R(C) = H(C)$ and $d \leq 2^{k+1}$;

iii) $q = 2$, $R(C) = H(C)-1$ and $d \leq 2^k$;

iv) $R(C_o^{(i)}) \leq R(C) +1$, for some i ;

v) $R(C) = H(C)$ and $d \leq q^k$.

In [35] Kilbay and Sloane proved that any binary [n,k,d] code s normal if either $n \leq 14$ or $k \leq 5$ or $d \leq 5$ However the case d=5 is not ue (X.Hou, IEEE Trans.Inform.Theory, pp.683-685,1990). They also gave amples of abnormal binary non-linear codes. At present it is not known an abnormal binary linear code exists.

E THE NUMBER $\max_s(r,q)$

A subset S of $GF(q)^r$ having n elements is called an ,s)-set if any s elements of S are linearly independent.

The largest value of n for which an (n,s)-set in $GF(q)^r$ exists is denoted by $\max_s(r,q)$ and a $(\max_s(r,q),s)$-set T is called complete. Determination of $\max_s(r,q)$ is called the packing problem, first considered by Bose [6] for statistical interest and later (Bose 1961) for its connection with coding theory.

Determination of $\max_s(r,q)$ in general is a difficult problem. It is known only in some special cases. A lower bound on $\max_s(r,q)$ can be obtained by considering codes. If C is an [n,n-r,d] code then any d-1 columns of a parity check matrix for C are linearly independent and so $\max_{d-1}(r,q) \geq n$. For $n > r$ the converse is also true.

For $s \geq 1$, $\max_s(r,q)$ is a strictly increasing function of r. To see this let $\max_s(r,q) = n$. Writing vectors of an (n,s)-set in $GF(q)^r$ as columns one gets an rxn matrix $H = [x_1, x_2, .., x_n]$ in which any s columns are linearly independent. Moreover any s columns of the matrix

$$H' = \begin{bmatrix} x_1 & x_2 & \cdots & x_n & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

are also linearly independent. Hence $\max_s(r+1,q) > n$. The following two propositions summarize some known results on $\max_s(r,q)$.

Proposition 2.23 [28;Ch.14]. (i) $\max_2(r,q) = (q^r-1)/(q-1)$,

(ii) $\max_3(3,q) = \begin{cases} q+1 \text{ if } q \text{ is odd} \\ q+2 \text{ if } q \text{ is even} \end{cases}$

(iii) $\max_3(4,q) = q^2+1$

(iv) If $\max_{d-1}(r-1,q) < n \leq \max_{d-1}(r,q)$ then $k_q[n,d] = n-r$, where $k_q[n,d] = \max\{k:$ there exists an $[n,k,d]$ code $\}$.

Proposition 2.24 [20]. For given r,q and d, suppose $\max_d(r+1,q) \leq n$, $\max_d(r,q) \leq m$ and $\max_d(r^\perp+1,q) \leq n$ where $r^\perp = n-r-2$, $d^\perp = n-m-1$. If $n > n^\perp$ then $\max_d(r+1,q) \leq n-1$.

Games [20] has found bounds on $\max_s(r,3)$ for $r \leq 15$, which were further improved by Hill [29]. For a survey of known results till 1983, concerning $\max_s(r,q)$ we refer to Hirschfeld [30].

CHAPTER III

COVERING RADIUS OF OPTIMUM CODES

III.A. A LOWER BOUND

Let C be a $[g(k,d), k, d]$ binary code with $d \leq 2^{k-1}$. Then columns of any generator matrix G for C are nonzero and distinct (Proposition 2.6). Hence C is a punctured binary Simplex code and so

$R(C) \geq R(S_k)$-number of columns deleted (Proposition 2.16)

$$= 2^{k-1} - 1 - (2^k - 1 - g(k,d)) = g(k,d) - 2^{k-1}.$$

Thus one has

Theorem 3.1. If C is a binary $[g(k,d), k, d]$ code with $d \leq 2^{k-1}$ then $R(C) \geq g(k,d) - 2^{k-1}$.

The best lower bound on the covering radius of a binary $[n,k]$ code is $t[n,k]$, the minimum covering radius of any binary $[n,k]$ code [8]. A table of lower and upper bounds for $t[n,k]$, $1 \leq k \leq n \leq 64$ has been given by Graham and Sloane [21].

There are Griesmer codes for which the lower bound given by Theorem 3.1 is better than $t[n,k]$, restricted to Griesmer codes. For example if C is a [62,6,31] once punctured, once shortened first order Reed-Muller code then

$g(k,d)-2^{k-1} = 30$, while $23 \leq t[62,6] \leq 27$. Table 3.1 below gives a list of Griesmer codes for which $g(k,d)-2^{k-1} \geq t[n,k]$. The existence of codes given in Table 3.1 follows from [2].

Table 3.1

k	d	n = g(k,d)	t[n,k]	$g(k,d)-2^{k-1}$
3	4	7	2	3
4	6	12	4	4
4	8	15	5	7
5	14	28	11	12
5	16	31	13	15
6	26	53	19-22	21
6	28	56	20-24	24
6	30	60	22-26	28
6	32	63	23-27	31

III.B. AN UPPER BOUND

Let C be an $[n,k,d]$ code, $n < n_q(k+1,d)$ and let $x \in GF(q)^n$ with $d(x,C) = R(C)$. If $R(C) \geq d+n+1-n_q(k+1,d)$ and if G is a generator matrix for C, then the matrix

$$\left[\begin{array}{c|c} \overleftarrow{1\ 1\ \cdots\ 1}\ n_q(k+1,d)-n-1\ \overrightarrow{} & x \\ \hline 0 & G \end{array}\right]$$

generates an $[n_q(k+1,d)-1,\ k+1,\ d]$ code. This contradicts the minimality of $n_q(k+1,d)$ and the following theorem is proved.

Theorem 3.2. The covering radius $R(C)$ of any $[n,k,d]$ code C with $n < n_q(k+1,d)$ satisfies $R(C) \leq d+n-n_q(k+1,d)$.

For any k and d, $n_q(k+1,d) > n_q(k,d)$; for if $n_q(k+1,d) \leq n_q(k,d)$, then an $[n_q(k+1,d)-1,\ k,\ d]$ code can be obtained from an $[n_q(k+1,d),\ k+1,\ d]$ code by deleting a suitable column from its parity check matrix, a contradiction. The number '$n_q(k+1,d)-n_q(k,d)$' is a useful parameter in determining an upper bound for the covering radius of an optimum code, and will be denoted by $b_q(k,d)$, i.e.,

$$b_q(k,d) = n_q(k+1,d) - n_q(k,d). \tag{3.1}$$

For simplicity one often writes $b(k,d)$ for $b_2(k,d)$. As shown above $b_q(k,d) \geq 1$. The following theorem shows that the multiplicity of a column in any generator matrix of an $[n_q(k,d),\ k,d]$ code cannot exceed $b_q(k-1,d)$.

Theorem 3.3. Let G be a generator matrix of an $[n_q(k,d),\ k,d]$ code. Then any column of G repeats atmost $b_q(k-1,d)$ times.

Proof. If some column of G repeats b' times, $b' > b_q(k-1,d)$, then using row and column operations, G can be put in the form

$$\left[\begin{array}{c|c} \overbrace{1\ 1\ \ldots\ 1}^{b'} & * * \ldots * \\ \hline 0 & G' \end{array}\right]$$

, where *'s are some elements in GF(q).

Clearly G' generates an [n, k-1, d] code ; $n < n_q(k-1,d)$, a contradiction to the minimality of $n_q(k-1,d)$.

Corollary 3.1. If $b_q(k-1,d) = 1$, then any generator matrix of an $[n_q(k,d), k,d]$ code is a submatrix of the generator matrix of $S_k(q)$.

If C is an optimum code, then Theorem 3.2 gives the following upper bound for its covering radius.

Corollary 3.2. The covering radius of an $[n_q(k,d), k, d]$ code C satisfies $R(C) \leq d-b_q(k,d)$.

A necessary and sufficient condition for equality in Corollary 3.2 is given by the following theorem.

Theorem 3.4. An $[n_q(k,d), k, d]$ code has covering radius $d-b_q(k,d)$ if and only if there exists an $[n_q(k+1,d), k+1, d]$ code with $b_q(k,d)$ equivalent coordinates.

Proof. Let C be an $[n = n_q(k,d), k, d]$ code with covering radius $d-b_q(k,d)$ and generator matrix G. Let $x \in GF(q)^n$ such that $d(x,C) = d-b_q(k,d)$. Then the matrix

$$G' = \left[\begin{array}{c|c} \overbrace{1\ 1\ .\ .\ .\ 1}^{b_q(k,d)} & x \\ \hline 0 & G \end{array}\right]$$

generates an $[n_q(k+1,d),\ k+1,\ d]$ code with $b_q(k,d)$ equivalent coordinates. Conversely if there exists an $[n_q(k+1,d),k+1,d]$ code C' with $b_q(k,d)$ equivalent coordinates then by row and column operations a generator matrix G' for C' can be put in the above form. Clearly G will generate an $[n_q(k,d),k,d]$ code with $d(x,C) \geq d-b_q(k,d)$. Since C is optimum , $R = d-b_q(k,d)$.

III.B.1. APPLICATIONS

1. The bound given by Corollary 3.2 is attained in many cases:

   (i) Let C be any binary [21,5,10] code. Then $b(5,10) = 2$ [2]. Since $t[21,5] = 8$[21], $R(C) = 8$.

   (ii) Let C be any binary [16,5,8] code and let C' be the [15,5,7] once punctured code C. Since $t[15,5] = 5$[21] and $b(5,7) = 2$[2], $R(C') = 5$. Hence by Proposition 2.16, $R(C) = 6$.

2. The upper bound given by Corollary 3.2 is helpful in determining the covering radius of MacDonald codes.

Theorem 3.5. (i) $R(C_{k;1}) = 2^{k-1}-2$;

(ii) $R(C_{k;2}) = 2^{k-1}-4,\ k \geq 4$;

(iii) $2^{k-1}-8 \leq R(C_{k;3}) \leq 2^{k-1}-7,\ k \geq 6$.

Proof. By Theorem 3.1, $R(C_{k;u}) \geq 2^k-2^u-2^{k-1} = 2^{k-1}-2^u$. Since $b(k,2^{k-1}-1) = 1$ [16] , $b(k,2^{k-1}-2) \geq 2$ for $k \geq 4$ [16] and $b(k,2^{k-1}-4) \geq 3$ for $k \geq 6$ [17; Lemma 4.2]. The

result follows from Corollary 3.2.

Corollary 3.3. $b(k,2^{k-1}-2) = 2$ and $b(k,2^{k-1}-4) \leq 4$.

By techniques similar to those used in Theorem 3.5 lower and upper bounds on $R(C_{k;u})$, for any u, $1 \leq u \leq k-1$ can be found, but they are far away from each other for $u \geq 4$. It will be interesting to determine $R(C_{k;u}(q))$, in general.

3. A sufficient condition for the normality of optimum codes can be determined using Theorem 3.2.

Let C be an $[n = n_q(k,d),k,d]$ code. Choose a codeword $(c_1,c_2,..,c_n)$ of weight d and a coordinate i with $c_i = o$. Recall that $C_o^{(i)}$ denotes the set of all codewords with ith coordinate zero. Let D be the code obtained from $C_o^{(i)}$ by deleting the ith coordinate. D is an [n-1, k-1, d] code. Moreover a parity check matrix H' for D can be obtained by deleting the ith column from a parity check matrix H of C. Since at least d-1 columns of H' are required to write the deleted column ; $R(D) \geq d-1$. So by Theorem 3.2 $R(D) = d-1$ and hence $R(C_o^{(i)}) = d$ (Proposition 2.16). This proves the following lemma.

Lemma 3.1. Let C be an optimum code with minimum distance d, then there exists a coordinate i such that $R(C_o^{(i)}) = d$.

If $R(C) = d-1$ or $d-2$, then the following two results follow from the above lemma and Proposition 2.22 ((i),(iv)).

Theorem 3.6. Optimum codes with covering radius $d-1$ are normal.

Theorem 3.7. Binary optimum codes with covering radius $d-1$ or $d-2$ are normal.

It may be observed that if $n(k,d) = g(k,d) + t$, $t \in \{0,1\}$, $d \leq 2^{k+1}$ ($q > 2$, $n_q(k,d) = g_q(k,d)+t$, $\lceil d/q^k \rceil = t+1$) then Theorem 3.7 (Theorem 3.6) is a corollary to Proposition 2.22 (ii) and (iii) (Proposition 2.22 (v)).

III.B.2. AN UPPER BOUND ON b(k,d)

In [1] Adams has shown that if C_o is any subcode of index two of a binary code C, then $R(C) \geq \lfloor R(C_o)/2 \rfloor$. Moreover if C is an optimum code with minimum distance d then by Lemma 3.1, there is a coordinate i such that $R(C_o^{(i)}) = d$ and $C_o^{(i)}$ is a subcode of index two. Thus

$$\left\lfloor \frac{d}{2} \right\rfloor \leq R(C) \leq d-b(k,d)$$

and the following theorem is proved.

Theorem 3.8. $b(k,d) \leq \lceil d/2 \rceil$.

So $n(k+1,d)$ cannot be arbitrarily away from $n(k,d)$. The above theorem can be improved further in some cases.

Theorem 3.9. If $k \geq 2$ and $d \geq 5$, then $b(k,d) \leq \lceil d/2 \rceil - 1$.

Proof. Let C be a binary $[n,k,d]$ optimum code. If d is odd and $b(k,d) = \lceil d/2 \rceil$, then by Corollary 3.2, $R(C) = \lfloor (d-1)/2 \rfloor$

and hence C is perfect. So C must be any one of the following three types

(i) $[2^m-1, 2^m-m-1, 3]$: Single error-correcting Hamming code;

(ii) $[23,12,7]$: Golay code ;

(iii) $[n,1,n]$: Repetition code.

But C cannot be of type (i) or (iii) as $k \geq 2$ and $d \geq 5$. So C must be a $[23,12,7]$ Golay code. But $b(12,7) = 3$ [44], a contradiction. Since $b(k,d-1) = b(k,d)$ for any even d, the assertion follows.

A sufficient condition for a binary optimum code to be quasiperfect is given by the following theorem.

Theorem 3.10. For given k and d if (i) $k \neq 1, 12$ (ii) $d=5$ or $d \geq 9$, d odd and (iii) $b(k,d) = \lceil d/2 \rceil - 1$, then any binary $[n(k,d),k,d]$ code C has covering radius $d-b(k,d)$ and is quasiperfect.

Proof. If $R(C) < d-b(k,d)$, then the code is perfect, a contradiction. Therefore $R(C) = d-b(k,d) = \lfloor (d-1)/2 \rfloor + 1$ and code is quasiperfect.

Since $b(4,5) = b(9,5) = b(14,5) = 2$ [44] , one has

Corollary 3.4. Binary $[11,4,5]$, $[17,9,5]$ and $[23,14,5]$ codes are quasiperfect.

III.B.3. PERFORMANCE

The bound given by Corollary 3.2 is better than the best known upper bound (Proposition 2.20).

Theorem 3.11. If $n_q(k,d) = g_q(k,d)$ or $d \leq q^k$ then $d-b_q(k,d) \leq d-\lceil d/q^k \rceil$. However if $n_q(k,d) = g_q(k,d)+t$ and $n_q(k+1,d) = g_q(k+1,d)+t_1$, $o \leq t < t_1$; then $d-b_q(k,d) < d - \lceil d/q^k \rceil$.

The bound given by Corollary 3.2 is best possible for $b_q(k,d) = 1$.

Theorem 3.12. If $b_q(k,d) = 1$, then there exists an $[n = n_q(k,d),k,d]$ code with covering radius d-1.

Proof. Let $r = n-k$. Then $\max_{d-1}(r,q) \geq n$. If $\max_{d-1}(r,q) = n$, then $\max_{d-1}(r,q) < n+1 \leq \max_{d-1}(r+1,q)$. So by Proposition 2.23 $k_q(n+1,d] = k$. Since $b_q(k,d) = 1$, $n_q(k+1,d) = n+1$ and so there exists an [n+1, k+1, d] code over GF(q). Thus $k_q[n+1,d] \geq k+1$, a contradiction. Therefore $\max_{d-1}(r,q) > n$. Let S be a set of $\max_{d-1}(r,q)$ r-tuples over GF(q), such that any d-1 of them are linearly independent and let H be an rxn matrix whose columns are any n elements in S. Then the code C having H as a parity check matrix is an [n,k,d] code over GF(q). As S contains more than n elements, there exists an r-tuple x in S which is not a linear combination of any d-2 or fewer columns of H. So the covering radius R of C must be greater

than or equal to d-1. Since C is optimum ; R = d-1.

By Theorems 2.1 and 3.12 one gets an alternative proof for the following corollary.

Corollary 3.5. $R(S_k) = 2^{k-1}-1$.

The bound given by Corollary 3.2 is not attained for b(k,d) = 3.

Example 3.1. Let C be the [32,6,16] first order Reed-Muller code. Then b(6,16) = 3 as n(7,16) = 35 [43]. The covering radius for C is 12 [5] and 12 < d-b(k,d).

It will be interesting to determine whether the bound given by Corollary 3.2 is reached for b(k,d) = 2 or not.

III.C. THE NUMBER $max_s(r,q)$ AND ITS RELATION WITH OTHER PARAMETERS

An (n,s)-set S of $GF(q)^r$ can be <u>completed</u> if there exists an $(max_s(r,q),s)$-set T with $S \subseteq T$. The following theorem shows that there are (n,s)-sets of $GF(q)^r$ which cannot be completed.

Theorem 3.13. Let C be an [n,k,d] code over GF(q) with $n < max_{d-1}(n-k,q)$ and $R(C) \leq d-2$. Then the set S of columns of any parity check matrix for C cannot be completed.

Proof. If $x \in GF(q)^{n-k} \setminus S$, then x can be written as a linear combination of some d-2 or fewer elements of S. So

$S \cup \{x\}$ is not an $(n+1,d-1)$-set and hence S cannot be completed.

A binary [15,5,7] code C with R(C) = 5[18] satisfies the hypothesis of Theorem 3.13.

A complete set need not be unique.

Example 3.2. Let C be a [21,5,10] code. Since $\max_9(16,2) = 21$, the set S consisting of columns of any parity check matrix of C is complete. As there are exactly two inequivalent [21,5,10] codes [42], S is not unique.

If for given k,d and q, $b_q(k,d)$ is large enough, the following theorem is useful in determining the value of $\max_{d-1}(r,q)$ for certain values of r.

Theorem 3.14. For given k,d and q, let $n_q(k,d) = n$ and let $r = n-k$. If $b_q(k,d) \geq 2$ then $\max_{d-1}(r+t,q) = n+t$ for $0 \leq t \leq b_q(k,d)-2$ and $\max_{d-1}(r+b_q(k,d)-1,q) > n+b_q(k,d)-1$.

Proof. Let C be an [n,k,d] code with $b_q(k,d) \geq 2$. By appending zero columns to a generator matrix for C one gets $[n+1,k,d],\ldots,[n+b_q(k,d)-2,k,d]$ codes. However $[n+1,k+1,d],\ldots,[n+b_q(k,d)-1,k+1,d]$ codes do not exist. So $\max_{d-1}(r+t,q) = n+t$ for $0 \leq t \leq b_q(k,d)-2$. Since an $[n+b_q(k,d),k+1,d]$ code exists, $\max_{d-1}(r+b_q(k,d)-1,q) > n+b_q(k,d)$.

Relations among various parameters are explored in next two theorems.

Theorem 3.15. Let $n = n_q(k,d)$ and let $n-k = r$. Then the following are equivalent

(i) $\max_{d-1}(r,q) = n$;

(ii) $k_q[n,d] = k_q[n+1,d] = k$;

(iii) $b_q(k,d) \geq 2$;

(iv) every $[n,k,d]$ code has covering radius $\leq d-2$.

Proof. (i) ==> (ii) Since $\max_s(r,q)$ is an strictly increasing function of r, $\max_{d-1}(r-1,q) < n$ and $\max_{d-1}(r+1,q) \geq n+1$. Applying Proposition 2.23 we get

$$k_q[n,d] = k_q[n+1,d] = k.$$

(ii) ==> (iii) If $b_q(k,d) < 2$ then $b_q(k,d) = 1$ and hence there exists an $[n+1,k+1,d]$ code over GF(q) . So $k_q[n+1,d] \geq k+1$, a contradiction.

(iii) ==> (iv) Follows by Corollary 3.2.

(iv) ==> (i) Since $n = n_q(k,d)$, $\max_{d-1}(r,q) \geq n$. If $\max_{d-1}(r,q) > n$, let S be a $(\max_{d-1}(r,q),d-1)$-set and let H be an rxn matrix, whose columns are any n elements from S. The code C having H as a parity check matrix is an $[n,k,d]$ code . Since $n < \max_{d-1}(r,q)$, there exists $x \in S$ which is not a linear combination of any d-2 or fewer columns of H. So $R(C) \geq d-1$, a contradiction.

Theorem 3.16. Let n,k and d be given and let $n-k = r$. Then the following are equivalent

(i) $n_q(k,d) = n$;

(ii) $d_q[n-1,k] < d_q[n,k] = d$;

(iii) $k_q[n-1,d] < k_q[n,d] = k$;

(iv) $\max_{d-1}(r-1,q)+1 < n \leq \max_{d-1}(r,q)$.

Proof. (i) ==> (ii) Obviously $d_q[n,k] \geq d$. If $d_q[n,k] = d' > d$, then there exists an $[n,k,d']$ code and hence $n \geq n_q(k,d')$. Note that $n_q(k,d+1) > n_q(k,d)$, for if $n_q(k,d+1) \leq n_q(k,d)$, then an $[n_q(k,d+1)-1, k, d]$ code can be obtained from an $[n_q(k,d+1), k, d+1]$ code by deleting a suitable column from its generator matrix, a contradiction to the minimality of $n_q(k,d)$. Hence

$$n \geq n_q(k,d') > n_q(k,d) = n,$$

a contradiction. Moreover $d_q[n-1,k] < d$, otherwise there exists an $[n-1, k, d]$ code, a contradiction to the minimality of $n_q(k,d)$.

(ii) ==> (iii) Obviously $k_q[n,d] \geq k$. If $k_q[n,d] > k$, then an $[n, k+1, d]$ code would exist which gives an $[n-1, k, d]$ code , a contradiction as $d_q[n-1,k] < d$. Again the nonexistence of an $[n-1, k, d]$ code implies $k_q[n-1,d] < k$.

(iii) ==> (iv) Since $k_q[n-1,d] < k_q[n,d] = k$, there exists an $[n, k, d]$ code. But an $[n-1, k, d]$ code does not exist. Therefore $\max_{d-1}(r,q) \geq n$ and $\max_{d-1}(r-1,q) < n-1$.

(iv) ==> (i) $n_q(k,d) \leq n$, as there exists an [n, k, d] code. Since $n-1 > \max_{d-1}(r-1,q)$, an [n-1, k, d] code does not exist. Hence $n_q(k,d) = n$.

Corollary 3.6. $n_q(k,3) = k+r$, where r satisfies

$$\frac{q^{r-1}-1}{q-1} - r + 1 < k \leq \frac{q^r-1}{q-1} - r \ , \ r \geq 2.$$

Proof. By Proposition 2.23 $\max_2(r,q) = \frac{q^r-1}{q-1}$ and so $\max_2(r-1,q) + 1 < k+r \leq \max_2(r,q)$. The result follows by Theorem 3.16.

Theorem 3.17. Let C be an [$n = n_q(k,3)$, k, 3] code . Then

$$R(C) = \begin{cases} 1 & \text{if } k = ((q^r-1)/(q-1)) - r \text{ for some } r \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $k = ((q^r-1)/(q-1))-r$ for some r then by Corollary 3.6 $n_q(k,3) = ((q^r-1)/(q-1))$. Let H be a parity check matrix for C. Then columns of H are nonzero and inequivalent. Hence $C = H_r(q)$ and so $R(C) = 1$. On the other hand if

$$\frac{q^{r-1}-1}{q-1} - (r-1) < k < \frac{q^r-1}{q-1} - r \text{ then } n_q(k,3) = \frac{q^r-1}{q-1} - t, \ t \neq o.$$

Since $\sum_{i=o}^{1} \binom{n}{i}(q-1)^i < q^{n-k}$, by (2.14) $R(C) \geq 2$. Since C is optimum , $R(C) = 2$.

In [29] Hill has shown that $13 \leq \max_6(9,3) \leq 20$ and $14 \leq \max_7(10,3) \leq 21$. These bounds are further improved by the following theorem.

Theorem 3.18. $14 \leq \max_6(9,3) \leq 19$ and $15 \leq \max_7(10,3) \leq 20$.

Proof. Since $\max_8(11,3) = 16$ [29], there exists a [16,5,9] code C over GF(3). By deleting columns from a generator matrix of C one gets [15,5,8] and [14,5,7] codes over GF(3). Hence $15 \leq \max_7(10,3)$ and $14 \leq \max_6(9,3)$. As $\max_6(8,3) = 11$ and $\max_8(11,3) = 16$[29] hypothesis of Proposition 2.24 is satisfied and hence $\max_6(9,3) \leq 19$. Similarly $\max_7(10,3) \leq 20$.

III.D. BOUNDS ON $R(S_k(q))$

If $q > 2$, then almost nothing is known about the covering radius of $S_k(q)$. In [32] Janwa has posed this as an open problem. Lower and upper bounds on $R(S_k(q))$ are determined in the present section.

III.D.1. UPPER BOUNDS ON $R(S_k(q))$

Observe that $S_k(q)$ is an optimum code. Hence by (2.15) the following upper bound for $R(S_k(q))$ is obtained.

Theorem 3.19. $R(S_k(q)) \leq q^{k-1}-1$.

The bound given by the above theorem is attained if q is even and k = 2.

Theorem 3.20. If q is even then $R(S_2(q)) = q-1$.

Proof. Let $q = 2^m$. By Proposition 2.12, $b_q(2,q) = 1$. Hence by Theorem 3.12 there exists a [q+1, 2, q] code with

covering radius q-1. Since every $[(q^k-1)/(q-1), k, q^{k-1}]$ code is $S_k(q)$, the assertion follows.

The bound given by Theorem 3.19 can be improved in some cases.

Theorem 3.21. If q is odd then $R(S_2(q)) \leq q-2$.

Proof. By Proposition 2.12 $n_q(3,q) = q+3$. Thus $b_q(2,q) = 2$ and the result follows using Corollary 3.2.

The bound given by Theorem 3.21 is attained.

Example 3.3. $S_2(5)$ is a [6,2,5] code over GF(5) with generator matrix

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is easy to verify that for $a = (2,3,4,4,1,1) \in GF(5)^6$, $wt(a + S_2(5)) = 3$ and hence $R(S_2(5)) = 3$.

Theorem 3.22. $R(S_k(3)) \leq 3^{k-1}-2$ for $k \geq 2$.

Proof. If $k \geq 2$ then $n_3(k+1,3^{k-1}) \geq g_3(k+1,3^{k-1})+1 \geq g_3(k,3^{k-1})+2$ (Proposition 2.13). Thus $b_3(k,3^{k-1}) \geq 2$ and hence by Corollary 3.2 $R(S_k(3)) \leq 3^{k-1}-2$.

The bound given by Theorem 3.22 is attained for $k = 2$ [32] and 3.

Example 3.4. $R(S_3(3)) = 7$.

$S_3(3)$ is a [13,3,9] code over GF(3) with generator matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

For $a = (1,2,2,1,1,2,1,2,1,2,1,2,2) \in GF(3)^{13}$, $wt(a+S_3(3)) = 7$.

Theorem 3.23. $R(S_k(4)) \leq 4^{k-1}-2$, $k \geq 3$.

Proof. If $k \geq 3$ then $n_4(k+1,4^{k-1}) \geq g_4(k+1,4^{k-1})+1 \geq g_4(k,4^{k-1})+2$ (Theorem 5.4). So $b_4(k,2^{k-1}) \geq 2$ and hence $R(S_k(4)) \leq 4^{k-1}-2$.

III.D.2. LOWER BOUND ON $R(S_k(q))$

Let $t_k(q)$ denote the maximum number of (-1)'s in any vector of $S_k(q)$ and let $\underline{1}$ denote the all one vector of length $n = (q^k-1)/(q-1)$; then $wt(\underline{1} + S_k(q)) = n-t_k(q)$. This gives the following lower bound for $R(S_k(q))$.

Theorem 3.24. $R(S_k(q)) \geq n-t_k(q)$.

A relation between $t_m(q)$ and $t_k(q)$ for $m > k$ is given by the following lemma.

Lemma 3.2. $t_m(q) = q^{m-k}\, t_k(q)$.

Proof. Let $GF(q) = \{0,1,\alpha_1,\alpha_2,..,\alpha_{q-2}\}$ and let $G_k(q)$ be a generator matrix of the Simplex code $S_k(q)$. Then

$$G_{k+1}(q) = \left[\begin{array}{c|c|c|c|c} 0\ 0\ \cdots\ 0 & 1 & 1\ 1\ \cdots\ 1 & \cdots & \alpha_{q-2}\ \alpha_{q-2}\cdots\alpha_{q-2} \\ G_k(q) & 0 & G_k(q) & \cdots & G_k(q) \end{array}\right]$$

is a generator matrix for $S_{k+1}(q)$ and $t_{k+1}(q)$ $= \max\ qt_k(q), ((q^k-1)/(q-1))+1$. If $t_k(q) \leq ((q^{k-1}-1)/(q-1))$, then $R(S_k(q) \geq n-t_k(q) \geq q^{k-1}$, a contradiction. So $qt_k(q) > (q^k-1)/(q-1)$ and hence $t_{k+1}(q) = qt_k(q)$. Repeating this (m-k) times one gets

$$t_m(q) = q^{m-k}t_k(q).$$

Two equivalent Simplex codes may have different values of $t_k(q)$. So one needs to determine the minimal value of $t_k(q)$ for given k and q to get a good lower bound on the covering radius.

Suppose for given q there exists a positive integer k_o and a vector $a = (a_1, a_2, \ldots, a_n)$ of weight n with

$$wt(a + S_{k_o}(q)) = R(S_{k_o}(q)).$$

Then multiplying the ith coordinate of $S_{k_o}(q)$ by a_i^{-1} for $1 \leq i \leq n$ one gets an equivalent code $S'_{k_o}(q)$ for which $t'_{k_o}(q)$, the maximum number of (-1)'s in any codeword of $S'_{k_o}(q)$, satisfies

$$n-t'_{k_o}(q) = wt(\underline{1} + S'_{k_o}(q)) = wt(a + S_{k_o}(q)) = R(S_{k_o}(q)).$$

Hence by Lemma 3.2 one has

Theorem 3.25. Let k_o be as defined above. Then
$R(S_k(q)) \geq n - q^{k-k_o} t'_{k_o}(q)$, for $k \geq k_o$

III.D.3. BOUNDS ON $R(S_k(q))$ for q = 3,4 and 5

If q = 3, then by Example 3.4, $t'_3(3) = n-R(S_3(3)) = 6$ and hence using Theorem 3.25 one gets the following lower bound for $R(S_k(3))$.

Theorem 3.26. $R(S_k(3)) \geq (2.3^{k-1}-3^{k-2}-1)/2$, for $k \geq 3$.

If $q = 2^m$ and there exists a vector a of weight n satisfying $wt(a + S_2(q)) = R(S_2(q))$. Then by Theorems and 3.25 one gets

(3.2) $R(S_k(q)) \geq ((q-1)q^{k-1}-(q-2)q^{k-2}-1)/(q-1)$, for $k \geq 2$ and $q=2^m$.

Theorem 3.27. $R(S_k(4)) \geq (3.4^{k-1}-2.4^{k-2}-1)/3$, for $k \geq 2$.

Proof. Let $a = (\omega,\omega,\omega^2,\omega^2,\omega)$, where $GF(4) = \{ o,1,\omega,\omega^2 \}$. It is easy to verify that $wt(a+S_2(4)) = 3 = R(S_2(4))$, where $S_2(4)$ has generator matrix

$$\begin{bmatrix} o & 1 & 1 & \omega & \omega^2 \\ 1 & o & 1 & 1 & 1 \end{bmatrix} .$$

The result follows from (3.2).

Theorem 3.28. $R(S_k(5)) \geq (3.5^{k-1}-2.5^{k-2}-1)/4$, for $k \geq 2$.

Proof. The assertion follows easily by Example 3.3 and Theorem 3.25.

CHAPTER IV

MINIMAL LENGTH OF 9-DIMENSIONAL BINARY LINEAR CODES

In [42] Tilborg has shown that if $d > \lceil (-3+\sqrt{8k+1})/2 \rceil 2^{k-1}$ then $n(k,d) = g(k,d)$. This bound is improved in Section A of the present chapter. Using the construction of binary Griesmer codes given by Helleseth [24] and the above imporovement the equality 'n(k,d) = g(k,d)' is shown to be true for $d \geq 258$ except for $324 \leq d \leq 336$. Bounds on n(9,d) for $d \leq 256$ are obtained first by using known results and then further improved by giving constructions or showing nonexistence of certain 9-dimensional binary linear codes. Note that if C is a binary [n,k,d] code with d odd then by adding an overall parity check one gets an [n+1, k, d+1] code. Thus $n(k,d+1) \leq n(k,d)+1$. But $n(k,d+1) > n(k,d)$, so $n(k,d) = n(k,d+1)-1$. Therefore one only needs to consider even values of d. Throughout this chapter an [n,k,d] code will mean a binary linear [n,k,d] code.

IV.A. n(9,d) FOR d> 256

Tilborg [42] has shown that if $d > \lceil (-3 + \sqrt{8k+1})/2 \rceil 2^{k-1}$ then $n(k,d) = g(k,d)$. In next two theorems this lower bound is improved. As defined earlier, let $s = \lceil d/2^{k-1} \rceil$ and let $s2^{k-1} = \sum_{i=1}^{p} 2^{u_i-1}$; $k > u_1 > u_2 > \ldots > u_p > 0$.

Theorem 4.1. $n(k,d) = g(k,d)$ for any $d > (t-1)2^{k-1}$, where t is the least positive integer satisfying $(t+1)(t+2) \geq 2k$, i.e., $t = \lceil (-3 + \sqrt{8k+1})/2 \rceil$.

Proof. It suffices to show that the condition given by Belov (Proposition 2.2) is satisfied. Let s, u_i's and p be as defined above. Then $s \geq t$ and

$$\sum_{i=1}^{t+1} u_i \leq \sum_{i=1}^{t+1} (k-i) = (t+1)k - \frac{(t+1)(t+2)}{2} \leq tk.$$

If $p-1 < t$ then $\sum_{i=1}^{\min\{p,s+1\}} u_i < pk \leq sk$. If $t \leq p-1 < s$ then

$$\sum_{i=1}^{\min\{p,s+1\}} u_i = \sum_{i=1}^{t+1} u_i + \sum_{i=t+2}^{p} u_i \leq tk + \sum_{i=t+2}^{p} (k-i) < (p-1)k < sk.$$

Also if $p-1 \geq s$ then

$$\sum_{i=1}^{\min\{p,s+1\}} u_i = \sum_{i=1}^{t+1} u_i + \sum_{i=t+2}^{s+1} u_i \leq tk + \sum_{i=t+2}^{s+1} (k-i) < sk.$$

Let t be as defined in Theorem 4.1. If $k \geq 9$ and $t = s+1$, let $v_1, v_2, \ldots, v_t$ be integers defined inductively as follows.

v_1 is the largest integer $u < k$ for which $\sum_{i=1}^{\min\{u-2,s+1\}} (u-i) \leq sk$. Suppose $v_1 > v_2 > \ldots > v_{j-1}$ are defined for some $j < t$. Choose v_j to be the largest integer $u < v_{j-1}$ for which

$$\sum_{i=1}^{\min\{u-2,s-j+2\}} (u-i) \leq sk - \sum_{i=1}^{j-1} v_i \cdot \tag{4.1}$$

Finally let

$$v_t = sk - \sum_{i=1}^{t-1} v_i \quad (4.2)$$

Observe that v_{t-1} is the largest integer satisfying (i) $v_{t-1} < v_{t-2}$ and (ii) $(v_{t-1}-3) \leq sk - \sum_{i=1}^{t-1} v_i$. So $0 < v_t < v_{t-1}$.

Theorem 4.2. $n(k,d) = g(k,d)$ for $d \geq (t-1)2^{k-1} - v$, where $v = 2^{v_1-1} + 2^{v_2-1} + \ldots + 2^{v_t-1} + 2^{v_t-2} + \ldots + 2$; $t, v_1, v_2, \ldots, v_t$ are as defined above.

Proof. If $d > (t-1)\,2^{k-1}$, then by Theorem 4.1 $n(k,d) = g(k,d)$. For $(t-1)2^{k-1} - v \leq d \leq (t-1)2^{k-1}$, $s = t-1$ and so

$$\sum_{i=1}^{p} 2^{u_i-1} = s2^{k-1} - d = (t-1)2^{k-1} - d \leq v.$$

If $\sum_{i=1}^{p} 2^{u_i-1} \geq \sum_{i=1}^{t} 2^{v_i-1}$, then $u_i = v_i$ for $1 \leq i \leq t$ and hence

$$\sum_{i=1}^{\min\{p,s+1\}} u_i = \sum_{i=1}^{t} v_i = sk.$$

If $\sum_{i=1}^{p} 2^{u_i-1} < \sum_{i=1}^{t} 2^{v_i-1}$, then either (i) there exist $1 \leq b_1 < b_2 < \ldots < b_p \leq t-1$ with $\sum_{i=1}^{p} 2^{u_i-1} = \sum_{i=1}^{p} 2^{v_{b_i}-1}$ or

(ii) there exist $1 \leq b_1 < b_2 < \ldots < b_j \leq t-1$ with

$\sum_{i=1}^{j-1} 2^{v_{b_i}-1} < \sum_{i=1}^{p} 2^{u_i-1} < \sum_{i=1}^{j} 2^{v_{b_i}-1}$. In case (i) holds $u_i = v_{b_i}$ for $1 \le i \le p$ and hence $\sum_{i=1}^{\min\{p,s+1\}} u_i = \sum_{i=1}^{p} v_{b_i} < sk$. In case (ii) holds $u_i = v_{b_i}$ for $1 \le i \le j-1$ and $u_{j-1+k} \le v_{b_j}-k$, for $k = 1,2,.., v_{b_j}-2$ and hence

$$\sum_{i=1}^{\min\{p,s+1\}} u_i \le \sum_{i=1}^{j-1} v_{b_i} + \sum^{\min\{v_{b_j}-2,\, s-j+2\}} (v_{b_j}-i) \le sk.$$

Thus in any case $\sum_{i=1}^{\min\{p,s+1\}} u_i \le sk$ and hence the result follows by Proposition 2.2.

In particular if $k = 9$, the above theorem gives

Corollary 4.1. $n(9,d) = g(9,d)$ for $d \ge 338$.

Proof. For $k = 9$; $t = 3$, $v_1 = 8$, $v_2 = 6$, $v_3 = 4$ and hence $v = 174$ and $(t-1)2^{k-1}-v = 338$.

If $d = 258$ or $314 \le d \le 322$, then by Proposition 2.2, $n(9,d) = g(9,d)$. On the other hand if $306 \le d \le 314$ then Helleseth and Tilborg [26] have shown that $n(9,d) = g(9,d)$. Hence the remaining cases for $d \ge 258$ are $260 \le d \le 304$ and $324 \le d \le 336$.

For constructing codes meeting the Griesmer bound for $k = 9$, $260 \le d \le 304$ we need the following construction of an anticode given by Helleseth [24].

Let C be an [n,k] code with generator matrix G and let

$$G' = [S_k|S_k|\ldots|S_k] \setminus G.$$

The code C' generated by G' is called an anticode of C [19]. On the other hand if G' is any matrix and if maximum multiplicity of any column of it is s then

$$G = \underbrace{[S_k|S_k|\ldots|S_k]}_{\text{s-copies}} \setminus G'$$

generates a code C having C', the code generated by G', as an anticode. Whenever G (or C) and G' (or C') are used they are related as above.

For any $k > u_1 > u_2 > \ldots > u_p > 0$, let $n' = \sum_{i=1}^{p} (2^{u_i}-1)$ and let $\mathcal{G}(u_1,u_2,\ldots,u_p)$ be the set of all kxn' matrices G' for which the distance between any two codewords of C' is atmost $\sum_{i=1}^{p} 2^{u_i-1}$. If $G' \in \mathcal{G}(u_1,u_2,..,u_p)$ and the corresponding G has rank k then Helleseth has shown that the code generated by G is a Griesmer code. Infact he proved the following

Proposition 4.1 [24]. If $G' \in \mathcal{G}(u_1,u_2,..,u_p)$ and if G generates a code C of dimension k then C is a Griesmer code with parameters $[s(2^k-1) - \sum_{i=1}^{p} (2^{u_i}-1)\ ,\ k,\ s.2^{k-1} - \sum_{i=1}^{p} 2^{u_i-1}]$.

This can be used to construct a [g(9,260), 9, 260] code. Note that g(9,260) = 524 and $2(2^9-1)-g(9,260) = 498 = \sum_{i=1}^{6} (2^{u_i}-1)$ with $u_1 = 8$, $u_2 = 7$, $u_3 = 6$, $u_4 = 5$, $u_5 = 4$, $u_6 = 3$. For each u, $1 \leq u \leq 9$, let $\hat{U}(9,u)$ be the set of all u-dimensional

subspace of $F(2)^9$ and let $U(9,u) = \{\hat{U} \setminus \{0\}: U \in U(9,u)\}$.

Let $V_1^{(1)} \ldots V_6^{(1)}$ be matrices defined by

$$V_1^{(1)} = \left[\begin{array}{c|c} 1\ 1\ \ldots\ 1 & 0\ 0\ \ldots\ 0 \\ \hline U_1 \setminus W_1 & W_1 \end{array}\right],$$

$U_1 \in U(8,8)$, $W_1 \in U(8,4)$, $W_1 \subseteq U_1$ and every element in W_1 has zeros at the last four positions,

$$V_2^{(1)} = \left[\begin{array}{c|c} 0\ 0\ \ldots\ 0 & 0\ 0\ \ldots\ 0 \\ 1\ 1\ \ldots\ 1 & 0\ 0\ \ldots\ 0 \\ \hline U_2 \setminus W_2 & W_2 \end{array}\right],$$

$U_2 \in U(7,7)$, $W_2 \in U(7,4)$, $W_2 \subseteq U_2$ and elements in W_2 have zeros at 2nd, 3rd and 4th positions,

$$V_3^{(1)} = \left[\begin{array}{c|c} 0\ 0\ \ldots\ 0 & 0\ 0\ \ldots\ 0 \\ 0\ 0\ \ldots\ 0 & 0\ 0\ \ldots\ 0 \\ 1\ 1\ \ldots\ 1 & 0\ 0\ \ldots\ 0 \\ \hline U_3 \setminus W_3 & W_3 \end{array}\right],$$

$U_3 \in U(6,6)$, $W_3 \in U(6,4)$, $W_3 \subseteq U_3$ and elements of W_3 have zeros at the last two positions,

$V_4^{(1)} \in U(9,5)$ such that $\hat{V}_4^{(1)}$ is generated by the vectors $(0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1)^{tr}$, $(1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1)^{tr}$, $(0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1)^{tr}$, $(0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0)^{tr}$ and $(0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)^{tr}$,

$V_5^{(1)} \in U(9,4)$ such that $\hat{V}_5^{(1)}$ is generated by the vectors $(0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0)^{tr}$, $(0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0)^{tr}$, $(0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)^{tr}$ and $(0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^{tr}$

and $V_6^{(1)} \in U(9,3)$ such that $\hat{V}_6^{(1)}$ is generated by the three vectors $(0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1)^{tr}$, $(0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0)^{tr}$ and $(0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0)^{tr}$.

If $G_1' = [V_1^{(1)}|V_2^{(1)}|V_3^{(1)}|V_4^{(1)}|V_5^{(1)}|V_6^{(1)}]$ then it is easy to check that $G_1' \in \mathcal{G}_4(8,7,6,5,4,3)$ and no column occurs more than twice. So $G_1 = [S_9|S_9] \setminus G_1'$ generates a [524,9,260] code (Proposition 4.1).

On replacing $\hat{V}_5^{(1)}$ and $\hat{V}_6^{(1)}$ by their respective subspaces one also obtains the following codes

Parameters of C_1	G_1' belongs to
[528,9,262]	$\mathcal{G}_4(8,7,6,5,4,2)$
[531,9,264]	$\mathcal{G}_4(8,7,6,5,4)$
[536,9,266]	$\mathcal{G}_4(8,7,6,5,3,2)$
[539,9,268]	$\mathcal{G}_4(8,7,6,5,3)$
[543,9,270]	$\mathcal{G}_4(8,7,6,5,2)$
[546,9,272]	$\mathcal{G}_4(8,7,6,5)$

Thus the following theorem is proved.

Theorem 4.3. If $260 \leq d \leq 272$ then $n(9,d) = g(9,d)$.

Now consider the case $273 \leq d \leq 288$. Define $V_1^{(2)}, V_2^{(2)}, \ldots, V_6^{(2)}$ by

$$V_1^{(2)} = \left[\begin{array}{c|c} 1\ 1\ \ldots\ 1 & 0\ 0\ \ldots\ 0 \\ \hline U_1 \setminus W_1 & W_1 \end{array}\right],$$

$U_1 \in U(8,8)$, $W_1 \in U(8,5)$, $W_1 \subseteq U_1$ and the vectors of W_1 have zeros at the last three positions,

$$V_2^{(2)} = \left[\begin{array}{c|c} o\ o\ \ldots.o & o\ o\ \ldots.o \\ 1\ 1\ \ldots.1 & o\ o\ \ldots.o \\ \hline U_2 \setminus W_2 & W_2 \end{array}\right],$$

$U_2 \in U(7,7)$, $W_2 \in U(7,5)$, $W_2 \subseteq U_2$ and the vectors of W_2 have zeros at 2nd and 3rd positions,

$$V_3^{(2)} = \left[\begin{array}{c|c} o\ o\ \ldots.o & o\ o\ \ldots.o \\ o\ o\ \ldots.o & o\ o\ \ldots.o \\ 1\ 1\ \ldots.1 & o\ o\ \ldots.o \\ \hline U_3 \setminus W_3 & W_3 \end{array}\right],$$

$U_3 \in U(6,6)$, $W_3 \in U(6,5)$, $W_3 \subseteq U_3$ and each element of W_3 has first entry zero,

$V_4^{(2)} \in U(9,4)$ such that $\hat{V}_4^{(2)}$ is generated by the vectors $(1\ 1\ 1\ o\ o\ o\ o\ o\ o)^{tr}$, $(o\ o\ o\ o\ o\ 1\ 1\ o\ o)^{tr}$, $(1\ o\ o\ o\ o\ o\ o\ o\ o)^{tr}$ and $(o\ o\ o\ 1\ 1\ o\ o\ o\ o)^{tr}$,

$V_5^{(2)} \in U(9,3)$ such that $\hat{V}_5^{(2)}$ is generated by the vectors $(o\ 1\ o\ o\ o\ o\ 1\ 1\ 1)^{tr}$, $(o\ 1\ 1\ o\ o\ o\ 1\ o\ o)^{tr}$ and $(1\ o\ o\ o\ o\ 1\ o\ 1\ 1)^{tr}$

and $V_6^{(2)} \in U(9,2)$ such that $\hat{V}_6^{(2)}$ is generated by the two vectors $(o\ o\ 1\ o\ 1\ 1\ 1\ 1\ o)^{tr}$ and $(1\ o\ o\ o\ 1\ 1\ 1\ o\ 1)^{tr}$.

If $G_2' = [\ V_1^{(2)} \mid V_2^{(2)} \mid V_3^{(2)} \mid V_4^{(2)} \mid V_5^{(2)} \mid V_6^{(2)}\ ]$ then $G_2' \in \mathcal{G}(8,7,6,4,3,2)$ and no column occurs more than twice. So the code generated by the matrix $G_2 = [S_9 | S_9] \setminus G_2'$ is a [552,9,274] Griesmer code. Deleting appropriate columns from $V_4^{(2)}$, $V_5^{(2)}$ and $V_6^{(2)}$ one also gets Griesmer codes for $276 \leq d \leq 288$.

Similarly if $V_1^{(3)},\ldots,V_6^{(3)}$ are defined by

$$V_1^{(3)} = \left[\begin{array}{cc|c} 1\ 1\ \ldots\ 1 & & 0\ 0\ldots 0 \\ \hline U_1 & W_1 & W_1 \end{array}\right],$$ $U_1 \in U(8,8)$, $W_1 \in U(8,6)$, $W_1 \subseteq U_1$, vectors of W_1 have zeros at the last two positions,

$$V_2^{(3)} = \left[\begin{array}{cc|c} 0\ 0\ \ldots\ 0 & & 0\ 0\ldots 0 \\ 1\ 1\ \ldots\ 1 & & 0\ 0\ldots 0 \\ \hline U_2 & W_2 & W_2 \end{array}\right],$$ $U_2 \in U(7,7)$, $W_2 \in U(7,6)$, $W_2 \subseteq U_2$, each element of W_2 has 4th entry zero,

$V_3^{(3)} \in U(9,5)$, vectors of $V_3^{(3)}$ have zeros at the 4th, 5th 6th and 7th positions,

$V_4^{(3)} \in U(9,4)$ such that $\hat{V}_4^{(3)}$ is generated by the vectors $(1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^{tr}$, $(1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0)^{tr}$, $(0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0)^{tr}$ and $(0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1)^{tr}$,

$V_5^{(3)} \in U(9,3)$ such that $\hat{V}_5^{(3)}$ is generated by the vectors $(1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0)^{tr}$, $(0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1)^{tr}$ and $(0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0)^{tr}$,

$V_6^{(3)} \in U(9,2)$ such that $\hat{V}_6^{(3)}$ is generated by the vectors $(1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0)^{tr}$ and $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)^{tr}$

and $G_3' = [V_1^{(3)}|V_2^{(3)}|V_3^{(3)}|V_4^{(3)}|V_5^{(3)}|V_6^{(3)}]$ then $G_3 = [S_9|S_9] \setminus G_3'$ generates a $[g(9,290),9,290]$ code . Deleting appropriate columns from $V_4^{(3)}$, $V_5^{(3)}$ and $V_6^{(3)}$ one gets Griesmer codes for $292 \leq d \leq 304$.

Hence the following theorem is proved.

Theorem 4.4. If $273 \leq d \leq 304$ then $n(9,d) = g(9,d)$.

IV.B. KNOWN BOUNDS ON $n(9,d)$ FOR $d \leq 256$

By Proposition 2.3, $n(9,d) = g(9,d)$ for $d \leq 256$ except for $3 \leq d \leq 126$, $131 \leq d \leq 188$ and $195 \leq d \leq 216$. For each of these cases $n(9,d) \geq g(9,d)+1$. Table 4.1 below gives known bounds on $n(9,d)$. The lower bound is obtained by showing nonexistence of certain codes, the Table I of Verhoeff [44] or Proposition 2.8. Upper bounds are obtained by (i) Table I of Verhoeff [44] for $3 \leq d \leq 58$ ($d[n,k] \geq d$ implies $n(k,d) \leq n$) ; (ii) Theorem 3.1 of [16] for $131 \leq d \leq 188$ and $195 \leq d \leq 216$. The values of $n(8,d)$ are taken from [14],[16],[17] and [27].

Table 4.1

d	g(9,d)	n(9,d)	d	g(9,d)	n(9,d)
4	13	14	22	48	50-52
6	17	18	24	51	53-55
8	20	21	26	56	58-60
10	25	27	28	59	61-63
12	28	30	30	63	65-67
14	32	34-35	32	66	69-70
16	35	38-39	34	73	74-77
18	41	43-45	36	76	77-80
20	44	46-48	38	80	82-85

d	g(9,d)	n(9,d)	d	g(9,d)	n(9,d)
40	83	85-87	80	162	$\geq$ 164
42	88	89-94	82	168	$\geq$ 169
44	91	93-96	84	171	$\geq$ 172
46	95	97-100	86	175	$\geq$ 177
48	98	100-102	88	178	$\geq$ 180
50	104	106-109	90	183	$\geq$ 185
52	107	109-113	92	186	$\geq$ 188
54	111	113-116	94	190	$\geq$ 192
56	114	116-118	96	193	$\geq$ 195
58	119	121-126	98	200	$\geq$ 202
60	122	$\geq$ 124	100	203	$\geq$ 205
62	126	$\geq$ 129	102	207	$\geq$ 209
64	129	$\geq$ 132	104	210	$\geq$ 212
66	137	$\geq$ 138	106	215	$\geq$ 217
68	140	$\geq$ 141	108	218	$\geq$ 220
70	144	$\geq$ 145	110	222	$\geq$ 224
72	147	$\geq$ 148	112	225	$\geq$ 227
74	152	$\geq$ 153	114	231	$\geq$ 233
76	155	$\geq$ 156	116	234	$\geq$ 236
78	159	$\geq$ 161	118	238	$\geq$ 240

d	g(9,d)	n(9,d)	d	g(9,d)	n(9,d)
120	241	$\geq$ 243	168	328	339-341
122	246	$\geq$ 248	170	343	344-346
124	249	$\geq$ 251	172	346	347-349
126	253	254	174	350	351-352
132	268	269	176	353	354-355
134	272	273	178	359	360-362
136	275	276	180	362	363-365
138	280	281-282	182	366	367-368
140	283	284-285	184	369	370-371
142	287	288-290	186	374	375
144	290	291-292	188	377	378
146	296	297-298	196	395	396
148	299	300-301	198	399	400
150	303	304	200	402	403
152	306	307	202	407	408
154	311	312-314	204	410	411
156	314	315-318	206	414	415-416
158	318	319-321	208	417	418-419
160	321	322-324	210	423	424
162	328	329-331	212	426	427
164	331	332-334	214	430	431
166	335	336-337	216	433	434

IV.C. UPPER BOUNDS ON n(9,d) FOR CERTAIN VALUES OF d

Let C be a binary [n, k, d] maximal code with $R(C) \geq R_o$ and generator matrix G. Then the matrix

$$\left[\begin{array}{c|c} x & \overbrace{1\ 1\ \ldots\ 1}^{d-R_o} \\ \hline G & 0 \end{array}\right]$$

generates an $[n+d-R_o,\ k+1,\ d]$ code where $x \in GF(q)^n$ such that $d(x,C) = R(C)$. This proves the following theorem.

Theorem 4.5. If C is a binary [n, k, d] maximal code with $R(C) \geq R_o$, then there exists an $[n+d-R_o,\ k+1,\ d]$ code.

If G is a generator matrix for an [n, k, d] code with first row having weight d then on deleting any i(<d) columns, which have a nonzero entry in the first row, one obtains an [n-i, k, d-i] code. This proves the following theorem.

Theorem 4.6. Any [n, k, d] code gives an [n-i, k, d-i] code for $i < d$.

Let C be the [128,8,64] Reed-Muller code. Then R(C) = 56[40 Applying Theorems 4.5 and 4.6 to C one gets [136,9,64],[134,9,62 and [132,9,60] codes.

Corollary 4.2. $n(9,64) \leq 136$, $n(9,62) \leq 134$ and $n(9,60) \leq 132$.

If C is an $[n = 128+n(7,d-64),8,d]$ code with $66 \leq d \leq 92$, then Dodunekov and Manev [16] have shown that there exists a

generator matrix G for C of the form

$$G = \left[\begin{array}{c|ccc} G_1 & 0 & 0 \ \ldots & 0 \\ & & G_2 & \end{array}\right]$$

where G_1 generates the [128,8,64] first order Reed-Muller code C_1 and G_2 generates an [n(7,d-64], 7, d-64] code C_2. Further if n(7,d-64) = n(8,d-64)-1, then C_2 can be chosen to have $R(C_2) = d-65$ (Theorem 3.12). As C is a catenation of C_1 and C_2, by Proposition 2.17, $R(C) \geq R(C_1) + R(C_2) = 56+d-65 = d-9$. Therefore by Theorem 4.5 there exists an [n+7,9,d] code. This proves the following Theorem.

Theorem 4.7. If n(8,d-64) = n(7,d-64)+1, $64 < d \leq 92$ then $n(9,d) \leq n(7,d-64) + 137$.

Corollary 4.3. $n(9,66) \leq 145$, $n(9,68) \leq 149$, $n(9,70) \leq 153$, $n(9,72) \leq 156$, $n(9,80) \leq 172$, $n(9,86) \leq 184$ and $n(9,88) \leq 187$.

Proof. From tables of Tilborg [43] and of Dodunekov and Manev [16] it is easy to see that n(8,d-64) = n(7,d-64)+1 for d = 66,68,70,72,80,86 and 88. Hence the assertion follows from Theorem 4.7.

Theorem 4.8. $n(9,100) \leq 216$.

Proof. Let C be a [203,8,100] code with generator matrix G of the form

$$G = \left[\begin{array}{c|c|c} G_1 & \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{matrix} \\ & G_2 & G_3 \end{array}\right],$$

where G_1, G_2, G_3 generate [128,8,64], [64,7,32] and [11,6,4] codes C_1, C_2 and C_3 respectively [16]. Since $R(C_1) = 56$[40], $R(C_2) = 28$[8] and C_3 can be chosen to have $R(C_3) = 3$, by Proposition 2.17

$$R(C) \geq R(C_1) + R(C_2) + R(C_3) = 87.$$

A [216,9,109] code can be constructed from C using Theorem 4.5.

The following upper bounds on n(9,d) for remaining values of d, $60 \leq d \leq 124$ are obtained by applying Theorem 4.6 to [184,9,86], [172,9,80],[216,9,100] and [254,9,126] codes.

Table 4.2

d	Upper bound on n(9,d)	d	Upper bound on n(9,d)	d	Upper boun on n(9,d)
74	166	96	212	112	240
76	168	98	214	114	242
78	170	102	230	116	244
82	180	104	232	118	246
84	182	106	234	120	248
90	206	108	236	122	250
92	208	110	238	124	252
94	210				

IV.D. NONEXISTENCE OF CERTAIN CODES

An improvement of the Griesmer bound for small distances was given by Dodunekov and Manev (Proposition 2.9). In case $k \geq 9$, the following theorem gives a further improvement.

Theorem 4.9. If $3 \leq i \leq k-4$, $k \geq 9$, then $n(k,2^{k-i}) \geq g(k,2^{k-i})+3$.

Proof. The assertion is proved by induction, first on k for $i = 3$ and then on i. Let $i = 3$. If $k = 9$ then by Table 4.1 $n(9,64) \geq g(9,64) +3$. Let $k > 9$. If C is a $[g(k,2^{k-3})+2,k,2^{k-3}]$ code, then $res(C,2^{k-3})$ is a $[g(k-1,2^{k-4}) +2 ,k-1,2^{k-4}]$ code, a contradiction to the induction hypothesis. Suppose $i > 3$ and that the statement is true for $i-1$ and any $k \geq 9$. For any $k > 9$, $n(k-1,2^{k-i}) = n(k-1,2^{k-1-(i-1)}) \geq g(k-1,2^{k-1-(i-1)}) +3$. Hence by Proposition 2.8 $n(k,2^{k-i}) \geq g(k,2^{k-i})+3$. If $k = 9$, then $i = 4$ or 5. By Table 4.1 $n(9,32) \geq g(9,32)+3$ and $n(9,16) \geq g(9,16)+3$.

Another lower bound for n(k,d) for certain values of d is given by the following theorem.

Theorem 4.10. Let $d = 2^{k-4}-2^m$, $k \geq 10$, $m \geq 0$. Then $n(k,d) \geq g(k,d)$

For the proof of the above theorem one needs the following four lemmas.

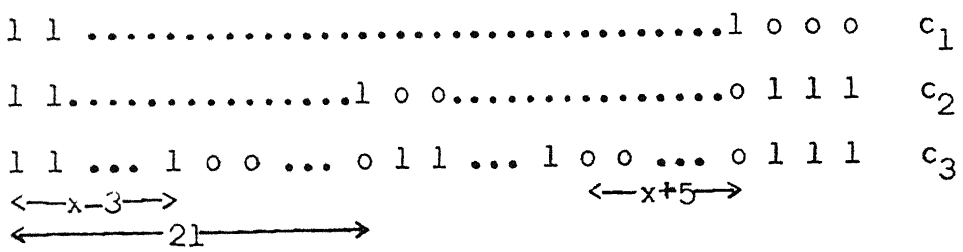
Lemma 4.1. $n(9,24) \geq 54$.

Proof. Suppose a [53,9,24] code C exists. Since $53 = g(9,24)+2$, by Proposition 2.4 there exists a generator matrix for C with rows having weights 24,25 or 26. But there is no vector of weight 25 in C. For, if $A_{25} \neq 0$ then res(C,25) is a [28,8,12] code which does not exist [16]. So possible nonzero weights in C are 24,32,34,36,40,44,48,50 and 52. Other A_i's are zero as the corresponding res(C,i) codes do not exist.

$A_{52} = 0$. For, if $c_1, c_2 \in C$ with $wt(c_1) = 52$ and $wt(c_2) = 24$, then it is easy to verify that $wt(c_1+c_2) = 28$ or 30, a contradiction.

If c_1 and c_2 are two vectors in C of weight w then $wt(c_1+c_2) \leq 2(53-w)$. Thus if $w \geq 42$, then $wt(c_1+c_2) \leq 22$. Therefore $A_i \in \{0,1\}$ for $i \geq 42$.

If $A_{50} = 1$ then $A_{34} = A_{36} = A_{40} = A_{44} = A_{48} = 0$. Let $c_1, c_2, c_3 \in C$, $wt(c_1) = 50$, $wt(c_2) = wt(c_3) = 24$. Without loss of generality (w. .o.g) one can assume c_1, c_2, c_3 have the following configuration.



So wt $(c_1+c_2+c_3) = 2x+2$ and $wt(c_2+c_3) = 48-2x$. This is not

possible for any x as only possible nonzero weights in C are 24,32,50. Hence $A_{50} = 0$.

$B_2 \leq 1$. For, if $B_2 \geq 2$, then any generator matrix for C can be put in one of the following forms

$$\left[\begin{array}{c|c} * \ * \ldots * & 1\ 1\ 1 \\ \hline G_1 & 0 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} \begin{matrix} * & * & \ldots & * \\ * & * & \ldots & * \end{matrix} & \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix} \\ \hline G_2 & \mathbf{0} \end{array}\right]$$

where *'s are some elements ,in GF(2).

But then G_1 will generate a [50,8,24] code and G_2 will generate a [49,7,24] code, a contradiction [44]. MacWilliams equations for B_0, B_1 and B_2 are

(4.3) $A_{24}+A_{32}+A_{34}+A_{36}+A_{40}+A_{44}+A_{48} = 511$

(4.4) $-5A_{24}+11A_{32}+15A_{34}+19A_{36}+27A_{40}+35A_{44}+43A_{48} = 53$

(4.5) $-7A_{24}+17A_{32}+23A_{34}+77A_{36}+169A_{40}+293A_{44}+449A_{48} = 256B_2-689.$

If $A_{48} \neq 0$, then $A_{48} = 1$, $A_{36} = A_{44} = 0$. A linear combination of (4.3) and (4.4) gives $4A_{32}+5A_{34} = 640$. So $A_{34} = 4k$ for some nonnegative integer k. Solving (4.3), (4.4) and (4.5) simultaneously one gets $1408+8k = 256B_2$, a contradiction as $B_2 \leq 1$. So $A_{48} = 0$.

If C has a codeword c_1 of weight 44, then $res(C,c_1)$ is a [9,8,2] code having a vector c_2^o of weight 4. Let $c_2 = (c_2^1 | c_2^o) \in C$. By Proposition 2.11

$$wt(c_1+c_2) = wt(c_1)+2wt(c_2^o)- wt(c_2) = 52 - wt(c_2),$$

which is not possible. Therefore $A_{44} = 0$.

Finally multiplying (4.3) by (-1/8),(4.4) by (-3/8), (4.5) by (1/4) and adding one gets

$$5A_{32} + 12A_{36} + 32A_{40} = -256 + 64B_2 \leq -192,$$

a contradiction. Hence $n(9,24) \geq 54 = g(9,24)+3$.

Corollary 4.4. $n(10,48) \geq 102 = g(10,48) +3$.

Proof. If C is any [101,10,48] code, then res(C,48) is a [53,8,24] code.

Lemma 4.2. Let $d = 2^m t (m \geq 2)$, 2\t and let C be an [n,k,d] code. If C has a generator matrix with all rows having weight d and if res(C,d) has all weights devisible by 2^{m-1}, then $wt(c) \equiv 0 \pmod{2^m}$ for all $c \in C$.

Proof. Let $c_1, c_2 \in C$, $wt(c_1) = wt(c_2) = d$. W.l.o.g. one can assume that c_1 and c_2 have the following configuration

$$c_1 = (1\ 1\ \ldots\ 1\ 0\ 0\ \ldots\ 0)$$
$$c_2 = (c_2^1 | c_2^0).$$

Since $c_2^0 \in res(C,d)$, $2^{m-1} | wt(c_2^0)$. Therefore $2^{m-1} | wt(c_1 * c_2)$ and hence $wt(c_1+c_2)$ is devisible by 2^m.

Dodunekov [13] has proved the nonexistence of a [116,9,56] code. An independent proof using the above lemma is given below.

Lemma 4.3. $n(9,56) \geq 117$.

Proof. Suppose there exists a [116,9,56] code C. Since $116 = g(9,56)+2$, by Proposition 2.4 C has a generator matrix with rows having weights 56,57 or 58. $A_{57} = 0$. For, if $A_{57} \neq 0$, then res(C,57) is a [59,8,28] code which does not exist [16]. Similarly $A_{58} = 0$. Res(C,56) is a [60,8,28] code. Since $60 = g(8,28) + 2$, res(C,56) has a generator matrix with rows having weights 28,29 or 30. But it cannot have a vector of weight 29 as then res(res(C,56),29) is a [31,7,14] code, which does not exist [43]. Therefore all weights in res(C,56) are even and hence by Lemma 4.2 all weights in C are devisible by 4. Thus possible nonzero weights in C are 56,64,68,80,96,112 and 116. $B_2 = 0$. For, if $B_2 \neq 0$, then by Proposition 2.10 there exists a [114,8,56] code, a contradiction as $n(8,56) = 115$ [14]. MacWilliams equations for B_0, B_1 and B_2 are

$$(4.6)\quad A_{56}+A_{64}+A_{68}+A_{80}+A_{96}+A_{112}+A_{116} = 511$$

$$(4.7)\quad -A_{56}+3A_{64}+5A_{68}+11A_{80}+19A_{96}+27A_{112}+29A_{116} = 29$$

$$(4.8)\quad -25A_{56}+7A_{64}+71A_{68}+455A_{80}+1415A_{96}+2887A_{112}+3335A_{116} = -33$$

If $A_{116} \neq 0$, then $A_{116} = 1$, $A_{64} = A_{68} = A_{80} = A_{96} = A_{112}=$ and hence (4.7) gives $A_{56} = 0$, a contradiction. Similarly if $A_{112} \neq 0$, then $A_{112} = 1$, $A_{68} = A_{80} = A_{96} = 0$. Solving (4.6)

and (4.7) simultaneously one gets $A_{56} = 382$, $A_{64} = 128$; but these values do not satisfy (4.8). Therefore $A_{116} = A_{112} = 0$.

If C has a vector of weight 96, then it is easy to verify that the [20,8,8] code $C^o = res(C,96)$ has unique weight distribution $A_0^o = 1$, $A_8^o = 130$, $A_{12}^o = 120$ and $A_{16}^o = 5$. Let $c_2 = (c_2 | c_2^o)$ C with $c_2^o \in C^o$. Then $wt(c_2^o) \in \{8,12,16\}$. Since $wt(c_1+c_2) = wt(c_1)+2wt(c_2^o)-wt(c_2)$, possible choices for $wt(c_2)$ for given value of $wt(c_2^o)$ are

$wt(c_2^o)$	8	12	16
$wt(c_2)$	56	56 or 64	64

Therefore $A_{68} = A_{80} = 0$ and $A_{96} = 1$. Solving (4.6) and (4.7) one gets $A_{56} = 380$, $A_{64} = 130$, which do not satisfy (4.8). Hence $A_{96} = 0$.

A linear combination of (4.6), (4.7) and (4.8) (with $A_{116} = A_{112} = A_{96} = 0$) gives

$$48A_{68} + 384A_{80} = 5120.$$

This is impossible as $3 \nmid 5120$. Hence a [116,9,56] code does not exist.

Lemma 4.4. $n(10,60) \geq 126$.

Proof. If $n(9,60) \geq 125$ then by Proposition 2.8, $n(10,60) \geq 12$ So one only needs to consider the case $n(9,60) < 125$. Let

C be a [124,9,60] optimum code. Proceeding on the same lines as in the proof of Lemma 4.3 it is easy to verify that all weights in C are divisible by 4. Thus possible nonzero weights in C are 60,64,72,76,88,104,120 and 124. $B_2 = 0$. For if $B_2 \neq 0$, then by Proposition 2.10 there exists a [123,8,60] code. But $n(8,60) \geq 123$[27]. MacWilliams equations for B_0, B_1 and B_2 are

$$(4.9) \quad A_{60}+A_{64}+A_{72}+A_{76}+A_{88}+A_{104}+A_{120}+A_{124} = 511$$

$$(4.10) \quad A_{60}+A_{64}+5A_{72}+7A_{76}+13A_{88}+21A_{104}+29A_{120}+31A_{124} = 31$$

$$(4.11) \quad 27A_{60}+27A_{64}-69A_{72}-165A_{76}-645A_{88}-1733A_{104}-3333A_{120} \\ -3813A_{124} = 3813.$$

$A_{124} = 0$. For, if $A_{124} \neq 0$, then $A_{124} = 1$, $A_{72} = A_{76} = A_{88} = A_{104} = A_{120} = 0$ and hence solving (4.9) and (4.10) one gets $A_{60} - A_{64} = 255$, but these values do not satisfy (4.11). With a similar reasoning $A_{120} = 0$.

If $A_{104} \neq 0$, then $A_{104} = 1$, $A_{88} = 0$ and hence solving (4.9) and (4.11) one gets

$$96A_{72} + 192A_{76} = 8224.$$

This is not possible as $3 \nmid 8224$. So $A_{104} = 0$.

If $A_{88} = 0$ and $A_{76} = 0$ then solving (4.9), (4.10) and (4.11) one gets $A_{64} = -41$, a contradiction. So either $A_{88} \neq 0$, or $A_{88} = 0$, $A_{76} \neq 0$. If $A_{88} \neq 0$ then $C^0 = \text{res}(C,88)$

is a $[36,8,16]$ code for which $b(8,16) \geq 2$ (Table 4.1). Hence by Corollary 3.2 $R(C^o) \leq 14$. Permuting columns, if necessary, any generator matrix for C can be put in the form

$$\left[\begin{array}{c|c} \overbrace{1\ 1 \ldots 1}^{88} & 0\ 0 \ldots 0 \\ \hline G' & G^o \end{array}\right],$$

G^o is a generator matrix for C^o. So by Proposition 2.18, $R(C) \leq \left[\frac{88}{2}\right] + R(C^o) \leq 58$.

On the other hand if $A_{88} = 0$, $A_{76} \neq 0$, then $res(C,76)$ is a $[48,8,22]$ code C^o for which $b(8,22) \geq 2$ (Table 4.1). Therefore by Corollary 3.2, $R(C^o) \leq 20$. Following the same procedure as above one gets $R(C) \leq \left[\frac{76}{2}\right] + R(C^o) \leq 58$. Thus in any case by Theorem 3.12, a $[125,10,60]$ code does not exist.

Proof of the Theorem 4.10. The assertion is proved using induction on m. If $k = 10$, then $0 \leq m \leq 5$. For $m = 0$, 1 or 5, Table 4.1 and Proposition 2.8 give $n(10,64-2^m) \geq g(10,64-2^m)+3$. If $m = 2$ or 4, then by Lemma 4.4 and Corollary 4.4 $n(10,64-2^m) \geq g(10,64-2^m) + 3$. Finally for $m = 3$, Proposition 2.8 and Lemma 4.3 give $n(10,56) \geq g(10,56)+3$.

Take $m > 0$, $k > 10$ and suppose the statement is true for m-1. Let C be a $[g(k,2^{k-4}-2^m)+2, k, 2^{k-4}-2^m]$ code. Then $res(C,d)$ is a $[g(k-1, 2^{k-5}-2^{m-1})+2, k-1, 2^{k-5}-2^{m-1}]$ code, which does not exist by the assumption.

Nonexistence of [195,9,96], [227,9,112] and [354,9,176] codes are shown in the next three theorems.

Theorem 4.11. $n(9,96) \geq 196$.

Proof. Suppose a [195,9,96] code C exists. Since 195 = $g(9,96) + 1$, by Proposition 2.4 there exists a generator matrix for C with rows having weights 96 or 97. $A_{97} = 0$. For if $A_{97} \neq 0$ then res(C,97) is a [98,8,48] code which does not exist [16]. So possible nonzero weights in C are 96,144,192 and 194. MacWilliams equations for B_0 and B_1 are

$$A_{96} + A_{144} + A_{192} + A_{194} = 511 \tag{4.12}$$

$$-3A_{96} + 93A_{144} + 189A_{192} + 193A_{194} = 195. \tag{4.13}$$

$A_{194} = 0$. For if $A_{194} \neq 0$, then $A_{194} = 1$, $A_{144} = A_{192} = 0$ and hence (4.13) gives $A_{96} = -2/3$, a contradiction . Similarly if $A_{192} \neq 0$, then $A_{192} = 1$, $A_{144} = 0$ and hence (4.13) leads to a false statement. Solving (4.12) and (4.13) simultaneously one gets $A_{96} = 493$, $A_{144} = 18$. Consider the [51,8,24] code C^o = res(C,144). It is easy to see that C^o has the unique weight distribution given by $A^o_0 = 1$, $A^o_{24} = 204$, $A^o_{32} = 51$. Let $c_1, c_2 \in C$, with $wt(c_1) = wt(c_2) = 144$. W.l.o.g. it can be assumed that c_1 and c_2 have the following configuration

$$\begin{array}{ll} 1\ 1 \ldots\ldots\ldots\ldots 1\ 0\ 0 \ldots\ldots\ldots\ldots 0 & c_1 \\ 1\ 1 \ldots 1\ 0\ 0 \ldots 0\ 1\ 1 \ldots 1\ 0\ 0 \ldots 0 & c_2 \end{array}$$

$\longleftarrow 48 \longrightarrow$

But then C^0 has a codeword of weight 48, a contradiction as $A^0_{48} = 0$. Thus C does not exist.

Theorem 4.12. $n(9,112) \geq 228$.

Proof. Suppose there exists a [227,9,112] code C. Proceeding on the same lines as in the proof of Lemma 4.3 it is easy to verify that all weights in C are devisible by 4. So possible nonzero weights in C are 112,128,176 and 224. MacWilliams equations for B_0, B_1 and B_2 are

(4.14) $A_{112}+A_{128}+A_{176}+A_{224} = 511$

(4.15) $-3A_{112}+29A_{128}+125A_{176}+221A_{224} = 227$

(4.16) $-1.9A_{112}+307A_{128}+7699A_{176}+24307A_{224} = -25651 + 512B_2$.

If $A_{224} \neq 0$, then $A_{224} = 1$, $A_{120} = A_{176} = 0$ and hence (4.14) and (4.15) do not have a common solution, a contradiction. So $A_{224} = 0$.

If $A_{176} \neq 0$, then $A_{176} = 1$. Solving (4.14) and (4.15) simultaneously one gets $A_{128} = 51$ and $A_{112} = 459$, but then (4.16) gives $B_2 = -2$, a contradiction. Hence $A_{176} = 0$.

Solving (4.14), (4.15) and (4.16) for B_2, one gets $B_2 = -14$, a contradiction. Therefore [227,9,112] code does not exist.

Theorem 4.13. $n(9,176) = 355$.

Proof. Suppose a [354,9,176] code C exists. Proceeding on

on the same lines as in the proof of Lemma 4.3 it is easy to verify that possible nonzero weights in C are 176, 192, 224, 228 and 352.

If C has a codeword c_1 of weight 352 and if c_2 is any codeword of weight 176 then permuting coordinates if necessary they must have the following configuration

$$1\ 1 \ldots\ldots\ldots\ldots\ldots 1\ 0\ 0\ c_1$$
$$1\ 1 \ldots 1\ 0\ 0 \ldots 0\ 0\ 0\ c_2$$

For, if the last two coordinates of c_2 are 10, 01 or 11 then $wt(c_1+c_2) = 178$ or 180, a contradiction. Since C has a generator matrix in which every row has weight 176, last two coordinates of C are identically zero and hence $n(C,176) \leq 352$, a contradiction to Table 4.1. Hence $A_{352} = 0$.

MacWilliams equations for B_0, B_1 and B_2 are

$$(4.17)\quad A_{176}+A_{192}+A_{224}+A_{228} = 511$$
$$(4.18)\quad -A_{176}+15A_{192}+47A_{224}+51A_{228} = 177$$
$$(4.19)\quad 175A_{176}-273A_{192}-4241A_{224}-5025A_{228} = 62481-512B_2$$

If $A_{228} \neq 0$, then $A_{228} = 1$. For, if $A_{228} \geq 2$, let $c_1, c_2 \in C$ with $wt(c_1) = wt(c_2) = 228$. Permuting coordinates, if necessary it can be assumed that they have the following configuration

$$1\ 1 \ldots\ldots\ldots\ldots\ldots 1\ 0\ 0 \ldots\ldots\ldots\ldots\ldots 0 \quad c_1$$
$$1\ 1 \ldots 1\ 0\ 0 \ldots 0\ 1\ 1 \ldots 1\ 0\ 0 \ldots 0 \quad c_2$$
$$\langle\!\!-\ x\ -\!\!\rangle \quad \langle\!\!-\ x\ -\!\!\rangle$$

Then $wt(c_1+c_2) = 2x$ and hence $x \in \{88,96,112,114\} = A$. Moreover if $c_2 = (c_2^1|c_2^0)$, then c_2^0 is a vector in $res(C,c_1)$ of weight x. So $res(res(C,c_1),x)$ is a $[126-x,7,62-[\frac{x}{2}]]$ code which does not exist for any $x \in A$[43]. Similarly $A_{224} = 0$. A linear combination of (4.17) and (4.18) give $16A_{192} = 636$, a contradiction. Therefore $A_{228} = 0$.

$A_{224} = 0$. For if, $A_{224} \neq 0$, then as above $A_{224} = 1$. Solving (4.17), (4.18) and (4.19) one gets $B_2 = -9$, a contradiction.

Thus $A_i = 0$ for $i \geq 224$ and the equations (4.17), (4.18) and (4.19) give $B_2 = -15$, a contradiction. So $n(9,176) \geq 355$. By Table 4.1, $n(9,176) \leq 355$. Therefore $n(9,176) = 355$.

CHAPTER V

OPTIMUM CODES OF DIMENSION 3 AND 4 OVER GF(4)

In [12] Dodunekov has shown that codes of dimension 1 and 2 meet the Griesmer bound. He also showed that a code of dimension 3 need not meet the Griesmer bound.The values of $n_4(3,d)$ and bounds on $n_4(4,d)$ are derived in this chapter by proving some general results.

V.A. GENERAL RESULTS

Let C be an [n, k, d] code and let $x \in C$ with $wt(x) = w$. If $w < d + [w/q]$ then res(C,x) is an $[n-w, k-1, d_o]$, $d_o \geq d-w + [w/q]$ code. So $n-w \geq k-1$ or $w \leq n-k+1$. This observation is useful in determining weight distribution of certain codes (Theorem 5.3) and is listed below as a lemma.

Lemma 5.1. If C is an [n, k, d] code having a vector of weight w, $w < d + [w/q]$ then $w \leq n-k+1$.

Proposition 2.8, proved by Dodunekov and Manev [16] can be generalized for codes over GF(q).

Theorem 5.1. Let $d \leq q^k$ and let $n_q(k,d) \geq g_q(k,d)+t$, for some positive integer t. Then $n_q(k+1,d) \geq g_q(k+1,d)+t$.

Proof. For $d \leq q^k$, $g_q(k+1,d) = g_q(k,d)+1$. Hence

$$n_q(k+1,d) \geq n_q(k,d)+1 \geq g_q(k,d)+t+1 = g_q(k+1,d)+t.$$

A relation between codes over $GF(p^m)$ and codes over $GF(p^s)$, s a divisor of m , is given by the following theorem.

Theorem 5.2. Let C be an [n, k, d] code over $GF(p^m)$ and let s be a positive divisor of m. Then there exists an $[n(p^m-1)/(p^s-1),\ tk,\ (p^s)^{t-1}d\ ]$ code over $GF(p^s)$, $t = m/s$.

Proof. Let $n' = (p^m-1)/(p^s-1)$. Note that n' is the length of the Simplex code $S_t(p^s)$ and t is the dimension of the vector space $GF(p^m)$ over the field $GF(p^s)$. Let $\{\alpha_1,\alpha_2,..,\alpha_t\}$ be a basis for $GF(p^m)$ and let $\beta_1,\beta_2,\ldots,\beta_t$ be the rows of a generator matrix of $S_t(p^s)$. Then the linear mapping f from $GF(p^m)$ into $GF(p^s)^{n'}$ satisfying $f(\alpha_i) = \beta_i$ is an isomorphism. Mapping each codeword $(x_1,x_2,\ldots,x_n) \in C$ to $(f(x_1),f(x_2),\ldots,f(x_n))$ one gets an $[n(p^m-1)/(p^s-1),tk,(p^s)^{t-}$ code over $GF(p^s)$.

The above theorem proves useful in (i) showing nonexistence of certain codes over $GF(p^m)$ (ii) the constructio of codes over $GF(p^m)$. This will be demonstrated in Sections V.C, V.D.

V.B. DETERMINATION OF $n_4(3,d)$

Let C be an $[n_4(3,d),3,d]$ code over GF(4). By (2.11) and (2.12) it is easy to see that C meets the Griesmer bound for all d except for d = 7 and 8. The following theorem shows that $n_4(3,7) > g_4(3,7)$.

Theorem 5.3. There does not exist a [10,3,7] code over GF(4).

Proof. Suppose there exists a [10,3,7] code C over GF(4). Then $A_0 = B_0 = 1$ and $B_1 = 0$. By Lemma 5.1 $A_9 = 0$. So possible nonzero weights in C are 7,8 and 10. MacWilliams equations [38] for B_0 and B_1 are

$$A_7 + A_8 + A_{10} = 63 \quad \text{and} \quad 3A_7 + 2A_8 = 150$$

respectively. Let c_1 and c_2 be any two codewords of weight 10. Then c_1 and c_2 are linearly dependent. For, if they are linearly independent, then by Lemma 2.13(i) of [29] $wt(c_1) + wt(c_2) \leq 19$, a contradiction. Therefore if $A_{10} \neq 0$ then $A_{10} = 3$ and hence $A_7 = A_8 = 30$, but then the MacWilliams equation for B_2 gives $B_2 = -15/4$, a contradiction. So $A_{10} = 0$ and one gets $A_7 = 24$, $A_8 = 39$ and $B_2 = -6$, a contradiction.

Corollary 5.1. $n_4(3,7) = 11$ and $n_4(3,8) = 12$.

Proof. By Proposition 2.14 and Theorem 5.3, $n_4(3,8) \geq 12$. Let C be a [15,6,8] code over GF(4) [10]. Since $15 = g_4(6,8)$ by Theorem 5.1, $n_4(3,8) \leq g_4(3,8)+1 = 12$ and hence $n_4(3,8) =$ Deleting any coordinate of a [12,3,8] code over GF(4), one gets an [11,3,7] code over GF(4). So $n_4(3,7) \leq 11$. Hence by Theorem 5.3 $n_4(3,7) = 11$.

V.C. BOUNDS ON $n_4(4,d)$

Theorem 5.4. If $k \geq 4$ and $d \in \{4^{k-2}-1, 4^{k-2}, 2.4^{k-2}-5, 2.4^{k-2}-4, 3.4^{k-2}-5, 3.4^{k-2}-4\}$ then $n_4(k,d) > g_4(k,d)$.

Proof. In view of Proposition 2.14 one needs to prove the inequality '$n_4(k,d) > g_4(k,d)$' for $d = 4^{k-2}-1, 2.4^{k-2}-5$, and $3.4^{k-2}-5$ only. Let $d = 4^{k-2}-1$ and let C be a $[g_4(k,d),k,d]$ code over GF(4). By Theorem 5.2 there exists a $[2^{2k-2}-1, 2k, 2^{2k-2}$ code C' over GF(2). But a binary code with these parameters does not exist as $d(2k, 2^{2k-2}-2) = 2^{2k-2}-2$ and $n(k, 2^{k-3}-2) \geq g(k, 2^{k-3}$ for $k \geq 7$ (Proposition 2.9).

If $d = 2.4^{k-2}-5$ and if C is a $[g_4(k,d), k, d]$ code over GF(4) then by Theorem 5.2 there exists a $[2^{2k-1}-17, 2k, 2^{2k-2}-10]$ binary code C' and hence $res(C', 2^{2k-2}-10)$ is a $[d(2k-1, 2^{2k-3} - 5) + 1, 2k-1, 2^{2k-3} - 5]$ code which does not exist for $k \geq 5$ [17; Lemma 4.2]. Also $res(C', 2^{2k-2}-10)$ code does not exist for $k = 4$ as $n(7,27) = g(7,27) + 2$ [43].

Finally, if $d = 3.4^{k-2}-5$ and if C is a $[g_4(k,d),k,d]$ code over GF(4) then there exists a $[3.2^{2k-2}-18, 2k, 3.2^{2k-3}-10]$ binary code C'. But a binary code with these parameters does not exist as $g(2k, 3.2^{2k-3}-10) = 3.2^{2k-2}-18$ and $n(k, 3.2^{k-3}-10) \geq g(k, 3.2^{k-3}-10) + 1$ for $k \geq 8$ (Proposition 2.3)

Theorem 5.5. If $d \in \{3,4,7,8,15,16,43,44\}$ or if $25 \leq d \leq 32$ then $n_4(4,d) > g_4(4,d)$.

Proof. If $d \in \{3,4,7,8,15,16,43,44\}$ then the inequality $n_4(4,d) > g_4(4,d)$ follows using Proposition 2.13 (for $d = 3,4$),

Corollary 5.1, Theorem 5.1 (for d = 7,8) and Theorem 5.4 (for d = 15,16,43,44). If a $[g_4(4,d),4,d]$ code C over GF(4) exists for $25 \leq d \leq 28$ ($29 \leq d \leq 32$) then res(C,d) is a [10,3,7 ([11,3,8]) code, a contradiction.

Putting q = 4 and k = 4 in Proposition 2.12 one gets the following theorem.

Theorem 5.6. $n_4(4,d) = g_4(4,d)$ for $45 \leq d \leq 48$, $57 \leq d \leq 64$, $93 \leq d \leq 96$, $105 \leq d \leq 112$, $117 \leq d \leq 128$, $141 \leq d \leq 144$, $157 \leq d \leq 160$, $173 \leq d \leq 192$, $201 \leq d \leq 208$ and $d \geq 213$. Moreover if the equality '$n_4(4,d) = g_4(4,d)$' holds for $d = 33+t_1$, $0 \leq t_1 \leq 7$ ($d = 9+t_2$, $0 \leq t_2 \leq 15$) then it also holds for $d = 161+t_1(137+t_2)$.

Since codes of dimension 2 meet the Griesmer bound, there exists a $[g_{16}(2,d),2,d]$ code over GF(4). Hence by Theorem 5.2 one gets the following result.

Theorem 5.7. There exists a $[5g_{16}(2,d),4,4d]$ code over GF(4).

Corollary 5.2. If $49 \leq d \leq 56$ or $113 \leq d \leq 116$, then $n_4(4,d) = g_4(4,d)$. Moreover $n_4(4,d) = g_4(4,d)+1$ for d = 43,44.

Proof. If d = 52,56 or 116 then $5g_{16}(2,d/4) = g_4(4,d)$. Thus by Proposition 2.14, $n_4(4,d) = g_4(4,d)$ for $49 \leq d \leq 56$ and $113 \leq d \leq 116$. Moreover $5g_{16}(2,11) = g_4(4,44)+1$. Deleting any coordinate of a $[g_4(4,44)+1,4,44]$ code over GF(4) one gets a $[g_4(4,43)+1,4,43]$ code over GF(4).

Table 5.1 below gives the lower and upper bounds on $n_4(4,$ for remaining values of d. The lower bounds are obtained by applying (2.2) and Theorem 5.5, while the upper bounds are obta using Theorem 5.7.

Table 5.1

d	$n_4(4,d)$	d	$n_4(4,d)$	d	$n_4(4,d)$
3	7-9	67	91-94	136	182-185
4	8-10	68	92-95	137	184-187
5	9-12	69	94-97	138	185-188
6	10-13	70	95-98	139	186-189
7	12-14	71	96-99	140	187-190
8	13-15	72	97-100	145	195-197
9	14-17	73	99-102	146	196-198
10	15-18	74	100-103	147	197-199
11	16-19	75	101-104	148	198-200
12	17-20	76	102-105	149	200-202
13	19-22	77	104-107	150	201-203
14	20-23	78	105-108	151	202-204
15	22-24	79	106-109	152	203-205
16	23-25	80	107-110	153	205-207
17	25-27	81	110-112	154	206-208
18	26-28	82	111-113	155	207-209
19	27-29	83	112-114	156	208-210
20	28-30	84	113-115	161	216-217
21	30-32	85	115-117	162	217-218
22	31-33	86	116-118	163	218-219
23	32-34	87	117-119	164	219-220
24	33-35	88	118-120	165	221-222
25	35-37	89	120-122	166	222-223
26	36-38	90	121-123	167	223-224
27	38-39	91	122-124	168	224-225
28	39-40	92	123-125	169	226-227
29	41-42	97	131-132	170	227-228
30	42-43	98	132-133	171	228-229
31	43-44	99	133-134	172	229-230
32	44-45	100	134-135	193	259-262
33	46-47	101	136-137	194	260-263
34	47-48	102	137-138	195	261-264
35	48-49	103	138-139	196	262-265
36	49-50	104	139-140	197	264-267
37	51-52	129	174-177	198	265-268
38	52-53	130	175-178	199	266-269
39	53-54	131	176-179	200	267-270
40	54-55	132	177-180	209	280-282
41	56-57	133	179-182	210	281-283
42	57-58	134	180-183	211	282-284
65	89-92	135	181-184	212	283-285
66	90-93				

V.D. SOME IMPROVEMENTS IN TABLE 5.1

The 4x8 matrix

$$G = \begin{bmatrix} 1 & o & o & o & o & 1 & 1 & 1 \\ o & 1 & o & o & 1 & o & 1 & 1 \\ o & o & 1 & o & 1 & 1 & o & 1 \\ o & o & o & 1 & 1 & 1 & 1 & o \end{bmatrix}$$

generates an [8,4,4] code over GF(4). By deleting any column of G one gets a [7,4,3] code over GF(4). Thus

Theorem 5.8. $n_4(4,3) = 7$ and $n_4(4,4) = 8$.

Theorem 5.9. $n_4(4,5) = 9\text{-}10$, $n_4(4,6) = 10\text{-}11$, $n_4(4,7) = 12$ and $n_4(4,8) = 13$.

Proof. Let C be a [15,6,8] code over GF(4) [10]. By Table 5.1 $13 \leq n_4(4.8) \leq 15$ and by Theorem 5.1 $n_4(4,8) \leq 13$. Hence $n_4(4,8) = 13$. By successively deleting a coordinate of a [13,4,8] code over GF(4) one gets [12,4,7], [11,4,6] and [10,4,5] codes over GF(4).

If C is a [51,8,24] code, then Dodunekov and Manev [16] have shown that C has a generator matrix G whose columns with numbers $\{i, i+17, i+34\}$ add upto zero for $1 \leq i \leq 17$. Let G_1 be the matrix obtained from G by permuting columns so that its columns with numbers $\{3i-2, 3i-1, 3i\}$ add up-to zero for $1 \leq i \leq 17$. Any three columns of G_1 with numbers $\{3i-2, 3i-1, 3i\}$; $1 \leq i \leq 17$ have the rows ooo, oll, lol or llo. If ω is a primitive element of GF(4), replacing ooo

by o,oll by 1,lol by ω and llo by ω^2 in G_1, one gets a [17,4,12] code C' over GF(4). By successively deleting a coordinate of C', the following theorem is obtained.

Theorem 5.10. $n_4(4,12-j) = 17-j$ **for** $0 \leq j \leq 3$.

Applying the above procedure to the 8x60 binary matrix

$$G_2 = \left[\begin{array}{c|cccccccc} G_1 & o & o & o & o & o & o & o & o \\ & o & o & o & o & o & o & o & o \\ & 1 & o & 1 & o & o & o & o & o \\ & o & 1 & 1 & o & o & o & o & o \\ & o & o & o & 1 & o & 1 & o & o \\ & o & o & o & o & 1 & 1 & o & o \\ & o & o & o & o & o & o & 1 & o \\ & o & o & o & o & o & o & o & 1 \end{array}\right],$$

where G_1 is as defined in the proof of Theorem 5.10, one gets a [20,4,13] code over GF(4). Thus

Theorem 5.11. $n_4(4,13) \leq 20$.

Theorem 5.12. $n_4(4,16-j) \leq 24-j$, $0 \leq j \leq 2$.

Proof. $S_3(4)$ is a [21,3,16] code over GF(4). By Theorem 3.27 $R(S_3(4)) = 13+t$, $t \geq 0$. So there exists $x \in GF(4)^{21}$ with $d(x,S_3(4)) = 13+t$ and the matrix

$$\left[\begin{array}{c|ccc} x & 1 & 1 & 1 \\ \hline G_3(4) & & 0 & \end{array}\right]$$

generates a [24,4,16] code C over GF(4). By successively deleting coordinates of C one gets [24-j,4,16-j] codes over GF(4) for $0 \leq j \leq 2$.

Using Theorems 5.6 and 5.10, existence of certain Griesmer codes over GF(4) is proved.

Theorem 5.13. If $137 \leq d \leq 140$ then any $[n_4(4,d),4,d]$ code over GF(4) is a Griesmer code.

Successively deleting coordinates of a $[g_4(4,137),4,137]$ code over GF(4) one gets the following result.

Theorem 5.14. (i) If $133 \leq d \leq 136$, then $n_4(4,d) \leq g_4(4,d)+1$.
(ii) If $129 \leq d \leq 132$, then $n_4(4,d) \leq g_4(4,d)+2$.

A summary of new bounds derived in Theorems 5.8-5.14 is given by the following table.

Table 5.2

d	$n_4(4,d)$	d	$n_4(4,d)$	d	$n_4(4,d)$
3	7	12	17	133	179-180
4	8	13	19-20	134	180-181
5	9-10	14	20-22	135	181-182
6	10-11	15	22-23	136	182-183
7	12	16	23-24	137	184
8	13	129	174-176	138	185
9	14	130	175-177	139	186
10	15	131	176-178	140	187
11	16	132	177-179		

REFERENCES

[1] M.J. Adams, ''Subcodes and Covering Radius'', IEEE Trans. Inform. Theory , Vol. IT-32, pp. 700-701,1986.

[2] L.D. Baumert and R.J. McEliece, ''A note on the Griesmer bound'', IEEE Trans. Inform. Theory, Vol. IT-19 pp. 134-135,1973.

[3] B.I. Belov, ''A conjecture on the Griesmer bound'', Optimization Methods and Their Applications, All-Union Summer Sem., Lake Baikal, 1972.

[4] B.I. Belov, V.N. Logachev and V.P. Sandimirov, ''Construction of a class of linear binary codes achieving the Varshamov-Griesmer bound'', Prob. Inform. Transm., Vol. 10, pp. 211-217,1974.

[5] E. Berlekamp and L.R. Welch, ''Weight distributions of the cosets of the (32,6) Reed-Muller code'', IEEE Trans. Inform. Theory, Vol. IT-18, pp. 203-207,1972.

[6] R.C. Bose, ''Mathematical theory of the symmetrical factorial design, ''Sankhya, Vol. 8, pp. 107-166,1947.

[7] P.B. Busschbach, M.G.L. Gerretzen and H.C.A van Tilborg, ''On the covering radius of binary, linear codes meeting the Griesmer bound'', IEEE Trans. Inform. Theory, Vol. IT-31, pp. 465-468,1985.

[8] G.D. Cohen, M.R. Karpovsky, H.F. Mattson Jr. and J.R. Schatz, ''Covering radius-Survey and recent results IEEE Trans. Inform. Theory, Vol. IT-31, pp. 328-343,1985

[9] G.D. Cohen, A.C. Lobstein and N.J.A. Sloane, ''Further results on the covering radius of codes'' , IEEE Trans. Inform. Theory, Vol. IT-32, pp. 680-694,1986.

[10] J.H. Conway, S.J. Lomonaco Jr. and N.J.A. Sloane, ''A [45,13] code with minimum distance 16'', Preprint.

[11] P. Delsarte, ''Four fundamental parameters of a code and their combinatorial significance'', Information and Control, Vol. 23, pp. 407-438,1973.

[12] S.M. Dodunekov, ''Minimal block length of a linear q-ary code with specified dimension and code distance'', Prob. Inform. Transm., Vol. 20, pp. 239-249,1984.

[13] S.M. Dodunekov, Proceedings of the fourth joint Swedish-Soviet International workshop in Information Theory, 27 Aug. to 1 Sep. 1989, Gotland, Sweden.

[14] S.M. Dodunekov, T. Helleseth, N.L. Manev and Ø. Ytrehu ''New bounds on binary linear codes of dimension eight IEEE. Trans. Inform. Theory, Vol. IT-33, pp. 917-919,1

[15] S.M. Dodunekov and N.L. Manev, ''Minimal possible bloc length of a linear binary code for some distances'', Prob. Inform. Transm., Vol. 20, pp. 8-14,1984.

[16] S.M. Dodunekov and N.L. Manev, ''An improvement of the Griesmer bound for some small minimum distances'', Discrete Applied Mathematics, Vol. 12, pp. 103-114,198

[17] S.M. Dodunekov and N.L. Manev, ''An improvement of the Griesmer bound for some classes of distances'', Prob. Inform. Transm., Vol. 23, pp. 38-46,1987.

[18] D.E. Downey and N.J.A. Sloane, ''The covering radius of cyclic codes of length upto 31'', IEEE Trans. Inform. Theory, Vol. IT-31, pp. 446-447,1985.

[19] P.G. Farrell, ''Linear binary anticodes, ''Electronics Letters, Vol. 6, pp. 419-421, 1970.

[20] R.A. Games, ''The packing problem for projective geometries over GF(3) with dimension greater than five Journal of Combinatorial Theory, Ser.A, Vol. 35, pp. 126-144,1983.

[21] R.L. Graham and N.J.A. Sloane, ''On the covering radius of codes'', IEEE Trans. Inform. Theory, Vol. IT pp. 385-401,1985.

[22] J.H. Griesmer, ''A bound for error-correcting codes'', IBMJ. Res. Develop., Vol. 4, pp. 532-542,1960.

[23] T. Helleseth, ''A characterization of codes meeting the Griesmer bound'', Information and Control. Vol. 50 pp. 128-159,1981.

[24] T. Helleseth, New constructions of codes meeting the Griesmer bound'', IEEE Trans. Inform. Theory, Vol. IT-29, pp. 434-439,1983.

[25] T. Helleseth, T. Kløve and J. Mykkeltviet, ''On the covering radius of binary codes'', IEEE Trans. Inform. Theory, Vol. IT-24, pp. 627-628,1978.

[26] T. Helleseth and H.C.A. van Tilborg, ''A new class of codes meeting the Griesmer bound'', IEEE Trans. Inform Theory, Vol. IT-27, pp. 548-555,1981.

[27] T. Helleseth and Ø.Ytrehus, ''New bounds on the minimum length of binary linear block codes of dimensi 8'', Reports in Informatics, Department of Informatics University of Bergen, Norway, Report No. 21,1986.

[28] R. Hill, A first course in coding theory, Clarendon, Oxford, 1986.

[29] R. Hill, Some optimal ternary linear codes'', Ars Combinatoria, Vol. 25A, pp. 61-72, June 1988.

[30] J.W.P. Hirschfeld, ''Maximum sets in finite projective spaces'', Surveys in Combinatorics, LMS Lecture Note Series 82, pp. 55-76,1983.

[31] H. Janwa, ''Relations among parameters of codes'', Ph.D. Thesis, Syracuse University, U.S.A., 1986.

[32] H. Janwa, ''Some new upper bounds on the covering radius of binary linear codes'', IEEE Trans. Inform. Theory, Vol. IT-35, pp. 110-122,1989.

[33] H. Janwa, ''Some optimal codes from algebraic geometry and their covering radii'', Presented at IMA Conference, June 1988, to appear in European Journal of Combinatorics.

[34] H. Janwa, ''On the covering radius of q-any codes'', to appear.

[35] K.E. Kilby and N.J.A. Sloave, ''On the covering radius problem for codes I and II'', Siam Journal Alg. Disc. Meth., Vol. 8, pp. 604-627, 1987.

[36] V.W. Logacev, ''An improvement of Griesmer bound in the case of small code distance, ''Optimization Methods and their Applications, All-Union Summer Sem., Lake Baikal, pp. 107-111,1972.

[37] V.N. Logachev, ''Characterization and existence conditi of codes meeting the Varshamov-Griesmer bound'' , Probl Peredachi Informatsii, Vol. 24, pp. 24-41,1988.

[38] F.J. MacWilliams and N.J.A. Sloane, The theory of error correcting codes, North Holland, Amsterdam,1981.

[39] H.F. Mattson Jr., ''An improved upper bound on covering radius'', Lecture Notes in Computer Science, 228, New Y Springer-Verlag, pp. 90-106,1986.

[40] J. Mykkeltviet, ''The covering radius of the (128,8) Reed-Muller code is 56'', IEEE Trans. Inform. Theory, Vol. IT-26, pp. 359-362,1980.

[41] G. Solomon and J.J. Stiffler, ''Algebraically punctured cyclic codes'', Information and Control, Vol.8, pp. 170-179,1965.

[42] H.C.A. van Tilborg, ''On the uniqueness resp. nonexistence of certain codes meeting the Griesmer bound'', Information and Control, Vol. 44, pp. 16-35,198

[43] H.C.A. van Tilborg, ''The smallest length of binary 7-dimensional linear codes with prescribed minimum distance'', Discrete Mathematics, Vol. 33,pp. 197-207,19

[44] T. Verhoeff, ''An updated table of minimum-distance bounds for binary linear codes",IEEE Trans. Inform. Theory, Vol. IT-33, pp. 665-680, 1987.

Date Slip

This book is to be returned on the date last stamped.
